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Pointwise estimates of solutions to the double-phase elliptic equations

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Abstract. With the help of nonlinear Wolf potentials, we derive the pointwise estimates for the weak solutions to inhomogeneous quasilinear double-phase elliptic equations of the divergence type.

Keywords. Wolf potential, double-phase quasilinear equation, equations of the divergence type, equations of the elliptic type, Harnack inequality.

1. Introduction

In the present work we obtain the pointwise estimates of the generalized solutions to inhomogeneous quasilinear elliptic equations of the divergence type

$$-\mathrm{div}\left(g(a(x), |\nabla u|)\frac{\nabla u}{|\nabla u|}\right) = f(x), \tag{1.1}$$

with the function $g(a(x), |\xi|) = |\xi|^{p-1} + a(x)|\xi|^{q-1}$ under the conditions

$$0 \le a(x) \in C^{0,\alpha}(\Omega),\ \alpha \in (0, 1],\ 1 < p \le q \le \min\left(p+\alpha,\ \frac{n(p-1)}{n-p}\right),$$

$q < n$.

Our result generalizes the classical one obtained by T. Kilpeläinen and J. Maly in [1]. With the help of nonlinear Wolf potential $W^{\mu}_{\beta,p}(x_0, R)$ they proved the pointwise estimates of solutions to a quasilinear elliptic equation with the p-Laplace and measure μ on the right-hand side. Further, these estimates were generalized to strongly nonlinear equations in [2] and to strongly nonlinear subelliptic quasilinear equations in [3] and were applied as an efficient tool to the study of the questions of solvability and solutions regularity to various linear, quasilinear and nonlinear equations (see the works of M. Biroli [4], F. Duzaar, J. Kristensen, and G. Mingione [5], J. Maly and W. Ziemer [6], G. Mingione [7], N. Phuc and I. Verbitsky [8], and I.I. Skrypnik [9]).

Due to application of some quasilinear equations with nonstandard growth conditions for the modeling of a behavior of electrorheological fluids (M. Ruzicka [10]), the qualitative theory of such equations is permanently developed, attracting the interest of researchers.

For example, for equations of the form

$$-\mathrm{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) + V\,|u|^{p(x)-2}u = f$$

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there were studied the questions of local regularity, the Harnack inequality was obtained and the Wiener criterion was proved under the natural assumptions on the function $p(x)$. The review of the corresponding results can be found in the papers of Y.A. Alkhutov [11], Y.A. Alkhutov and O.V. Krasheninnikova [12], X. Fan and D. Zho [13], V. Liskevich and I.I. Skrypnik [14].

On the other hand, the examples constructed by M. Giaquinta [15] and P. Marcellini [16] show that under conditions on function $g(t)$

$$t^{p-1} \le g(t) \le t^{q-1},$$

there exists unbounded solution (if p and q are too far from each other). For

$$q \le \frac{np}{n-p},\ 1 < p < n$$

and functions $g(t)$, satisfying condition above, the local properties of solutions have been studied in [17–33], if For the equations

$$-\mathrm{div}\left(g(|\nabla u|)\frac{\nabla u}{|\nabla u|}\right) = f(x)$$

under conditions

$$p-1 \le \frac{g'(t)}{g(t)} \le q-1,\ f \in L^s,\ s > n$$

the local boundedness of solutions and Hölder continuity were established in the work of G. Lieberman [34], also there was proved Harnack inequality. These results were generalized by many researchers (see, e.g., [7, 17–21, 23, 28] and [27]).

It is natural to assume that for the equations

$$-\mathrm{div}\left(g(a(x), |\nabla u|)\frac{\nabla u}{|\nabla u|}\right) = f(x)$$

with coefficients for which the Wolf potentials are finite, the Harnack inequality will be valid. The main difficulty in the proof of pointwise estimates consists in that E. De Giorgi [35] and J. Moser [36] techniques cannot be applied. We will use the iteration method developed in [1] for the p-Laplace operator. Applying this technique for our case, we obtain two-sided pointwise estimates for the generalized solutions of quasilinear double-phase elliptic equations of the divergence type.

2. Formulation of the main results

In a bounded domain $\Omega \subset \mathbb{R}^n$, $n \ge 2$ we consider an inhomogeneous quasilinear elliptic equation of the divergence type

$$-\mathrm{div} A(x,\, \nabla u) = f(x) \ge 0, \tag{2.1}$$

where $f(x) \in L^1(\Omega)$. We assume that the function $A(x,\, \xi) : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the conditions

1) $A(x,\, \xi)$ satisfies the Carathéodory condition,
2) $A(x, \xi)\xi \ge \mu_1(|\xi|^p + a(x)|\xi|^q)$,
3) $|A(x, \xi)| \le \mu_2(|\xi|^{p-1} + a(x)|\xi|^{q-1})$,

with some constants μ_1, $\mu_2 > 0$. We also assume that

$$0 \le a(x) \in C^{0,\,\alpha}(\Omega), \quad \alpha \in (0,\, 1],$$

$$1 < p \le q \le \min\left(p+\alpha,\, \frac{n(p-1)}{n-p}\right), \quad q < n. \tag{2.2}$$

The equations of the form (1.1) in which $g(a(x),\, t) = |t|^{p-1} + a(x)|t|^{q-1}$ can serve as examples of equations (2.1) under conditions 1)–3).

Let us introduce the necessary definitions.

Definition 2.1. *Let $G(a(x),t) = t(t^{p-1} + a(x)t^{q-1})$. Then $W^{1,G}(\Omega)$ denotes the class of functions u that are weakly differentiable in Ω and satisfy the condition*

$$\int\limits_{\Omega} G(a(x),\, |\nabla u|)\, dx < \infty.$$

Definition 2.2. *We say that u is a weak solution to Eq. (2.1), if $u \in W^{1,G}(\Omega)$ and it satisfies the integral identity*

$$\int\limits_{\Omega} A(x,\, \nabla u)\nabla\varphi\, dx = \int\limits_{\Omega} f\, \varphi\, dx, \tag{2.3}$$

for all $\varphi \in \overset{0}{W}{}^{1,G}(\Omega)$.

We will prove pointwise estimates for a nonnegative weak solution to the double-phase equation (2.1) in terms of the nonlinear Wolf potentials:

$$W^f_{1,p}(x_0,\, R) = \sum_{j=0}^{\infty} \left(\rho_j^{p-n} \int\limits_{B_{\rho_j}(x_0)} f\, dx \right)^{\frac{1}{p-1}},\ \rho_j = \frac{R}{2^j},\ j = 0,\, 1,\, ...$$

$$W^f_{1,q}(x_0,\, R) = \sum_{j=0}^{\infty} \left(\rho_j^{q-n} \int\limits_{B_{\rho_j}(x_0)} f\, dx \right)^{\frac{1}{q-1}},\ \rho_j = \frac{R}{2^j},\ j = 0,\, 1,\, ...,$$

under assumption that the series in the above formulae are convergent, i.e. the Wolf potentials are finite.

The main result of the present work is the following theorem.

Theorem 2.1. *Let $u \in W^{1,G}(\Omega) \cap L^{\infty}$ be a nonnegative weak solution to Eq. (2.1). Let conditions (2.2) be satisfied and let $[a]_{C^{0,\alpha}(\Omega)} := \sup\limits_{x,y\in\Omega,\, x\neq y} \frac{|a(x)-a(y)|}{|x-y|^{\alpha}}$. Assume also that the point $x_0 \in \Omega$ is such that $B_{4\rho}(x_0) \subset \Omega$. Then there exist constants $c_1,\, c_2 > 0$ depending only on $p,\, q,\, n,\, [a]_{C^{0,\alpha}(\Omega)}$ and $||u||^{q-p}_{L^{\infty}(\Omega)}$ such that, under condition $a(x_0) = 0$ the following estimate holds:*

$$c_1 W^f_{1,p}(x_0, \rho) \le u(x_0) \le c_2 \inf_{B_\rho(x_0)} u + c_2 W^f_{1,p}(x_0, 2\rho). \tag{2.4}$$

If $a(x_0) > 0$ and $\rho_0^{\alpha} = \frac{a(x_0)}{4[a]_{C^{0,\alpha}(\Omega)}} \ge \rho^{\alpha}$, then there exist constants $c_3,\, c_4 > 0$ depending on $p,\, q,\, n,\, [a]_{C^{0,\alpha}(\Omega)},\, ||u||^{q-p}_{L^{\infty}(\Omega)}$ and $a(x_0)$ such that the following estimate

$$c_3 W^f_{1,q}(x_0, \rho) \le \rho + u(x_0) \le 3\rho + c_4 \inf_{B_\rho(x_0)} u + c_4 W^f_{1,q}(x_0, 2\rho) \tag{2.5}$$

holds.

Under conditions $a(x_0) > 0$ and $\rho_0 < \rho$ will be true the estimate

$$c_3 W^f_{1,q}(x_0,\rho) + c_3(W^f_{1,p}(x_0,\rho) - W^f_{1,p}(x_0,\rho_0)) \le \rho + u(x_0)$$

$$\le 3\rho + c_4 \inf_{B_\rho(x_0)} u + c_4 W^f_{1,q}(x_0, 2\rho) + c_4(W^f_{1,p}(x_0,2\rho) - W^f_{1,p}(x_0,2\rho_0)). \tag{2.6}$$

Remark 2.1. In the case $a(x_0) = 0$ inequality (2.4) yields the known result of Kilpeläinen and Maly [1], where there were obtained the pointwise estimates of solutions to a quasilinear elliptic equation with the p-Laplace and measure μ on the right-hand side with the help of the nonlinear Wolf potential $W^{\mu}_{\beta,\,p}(x_0,\, R)$:

$$W^{\mu}_{\beta,\,p}(x_0,\, R) := \sum_{j=0}^{\infty} \left(\frac{\mu(B_{\rho_j}(x_0))}{\rho_j^{n-\beta p}} \right)^{\frac{1}{p-1}},\ \rho_j = \frac{R}{2^j},\ j = 0,\, 1,\, 2, ... \tag{2.7}$$

Let us note that double-phase elliptic equations of the divergence form were studied in first in the papers [37, 38] as models of strictly anisotropic materials and for the description of Lavrent'ev phenomenon. Hölder continuity and Harnack inequality for bounded solutions to the homogeneous equation (2.1) (with function $f \equiv 0$) were obtained in [17], [19] under conditions (2.2).

Theorem 2.1 is a consequence of the weak Harnack inequality (see, e.g., [17]) and the following result.

Theorem 2.2. *Let $u \in W^{1,G}(\Omega) \cap L^\infty$ be a nonnegative weak solution to Eq. (2.1). Let conditions (2.2) be satisfied and let the point $x_0 \in \Omega$ be such that $B_{4\rho}(x_0) \subset \Omega$.*

Let also $0 < \lambda < \min\left\{1,\ \frac{p(n-1)-q(n-p)}{n+(q-p)(n-p)}\right\}$. Then under condition $a(x_0) = 0$, the following estimate holds:

$$u(x_0) \le \gamma \left(\rho^{-n} \int\limits_{B_\rho(x_0)} u^{(1+\lambda)(p-1)}\, dx \right)^{\frac{1}{(1+\lambda)(p-1)}} + \gamma W^f_{1,p}(x_0,\rho). \tag{2.8}$$

If $a(x_0) > 0$ and $\rho_0^\alpha = \frac{a(x_0)}{4[a]_{C^{0,\alpha}(\Omega)}} \ge \rho^\alpha$, then the estimate

$$u(x_0) \le \gamma \left(\rho^{-n} \int\limits_{B_\rho(x_0)} u^{(1+\lambda)(q-1)}\, dx \right)^{\frac{1}{(1+\lambda)(q-1)}} + \gamma W^f_{1,q}(x_0,\rho) \tag{2.9}$$

holds.

Under conditions $a(x_0) > 0$ and $\rho_0 < \rho$ the following estimate is valid:

$$u(x_0) \le \gamma \left(\left(\rho^{-n} \int\limits_{B_\rho(x_0)} u^{(1+\lambda)(q-1)}\, dx \right)^{\frac{1}{(1+\lambda)(q-1)}} \right.$$

$$\left. + W^f_{1,q}(x_0,\, 2\rho_0) + (W^f_{1,p}(x_0,\, 2\rho) - W^f_{1,p}(x_0,\, 2\rho_0)) \right). \tag{2.10}$$

Here γ is some constant depending on μ_1, μ_2, p, q, n, $[a]_{C^{0,\alpha}(\Omega)}$, $||u||^{q-p}_{L^\infty(\Omega)}$.

Analogous result can be also obtained for a nonnegative weak solutions to the equations

$$-\mathrm{div}\left(g(|\nabla u|)\frac{\nabla u}{|\nabla u|}\right) = f(x) \geq 0, \tag{2.11}$$

with function $g(t)$ satisfying the conditions

$$g \in C(\mathbb{R}^1_+),\ \left(\frac{t}{\tau}\right)^{p-1} \leq \frac{g(t)}{g(\tau)} \leq \left(\frac{t}{\tau}\right)^{q-1},\ t \geq \tau > 0,\ 1 < p \leq q < n. \tag{2.12}$$

Harnack-type inequality can be also proved in terms of nonlinear Wolf potential $W^f_{\beta,g}(x_0, R)$:

$$W^f_{\beta,g}(x_0, R) := \sum_{j=0}^{\infty} \rho_j \bar{g}\left(\rho_j^{\beta-n} \int\limits_{B_{\rho_j}(x_0)} f\,dx\right),\ \rho_j = \frac{R}{2^j},\ j = 0,\ 1,\ 2, \ldots. \tag{2.13}$$

Here the function $\bar{g}$ denotes the inverse function for g.

The main result in this direction is the following theorem.

Theorem 2.3. *Let u be a nonnegative weak solution to Eq. (2.11), $f \geq 0$ and conditions (2.12) be satisfied. Then there exist constants c_5, $c_6 > 0$ depending only on p, q, n, μ_1, μ_2 such that, for every point $x_0 \in \Omega$, $B_{4\rho}(x_0) \subset \Omega$ the following estimates hold:*

$$c_5 W^f_{1,g}(x_0, \rho) \leq u(x_0) \leq c_6 \inf_{B_{\rho(x_0)}} u + c_6 W^f_{1,g}(x_0, 2\rho). \tag{2.14}$$

Here the Wolf potentials $W^f_{1,g}$ are defined by formula (2.13) for $\beta = 1$.

3. Upper bound of a solution, proof of Theorem 2.2

3.1. Auxiliary assertions

First of all we will prove some auxiliary assertions.

Lemma 3.1. *Let $0 < \lambda < 1$. Then for every weak solution u to Eq. (2.1), any $0 < l$, δ, $k \geq q$ and function $\xi \in C_0^\infty(B_r(x_0))$ satisfying the conditions*

$$0 \leq \xi \leq 1,\ \xi(x) \equiv 1,\ \forall\, x \in B_{\frac{r}{2}}(x_0),\ |\nabla \xi| \leq \frac{2}{r},$$

the following estimate is valid:

$$\int\limits_L \left(1 + \frac{u-l}{\delta}\right)^{-1-\lambda} |\nabla u|^p \xi^k\,dx$$

$$\leq \gamma\left(\left(\frac{\delta}{r}\right)^p + \left(\frac{\delta}{r}\right)^q\right) \int\limits_L \left(1 + \frac{u-l}{\delta}\right)^{(1+\lambda)(q-1)} \xi^{k-q}\,dx + \gamma\delta \int\limits_{B_r(x_0)} f\,dx, \tag{3.1}$$

if $a(x_0) = 0$ or $a(x_0) > 0$, $r > \rho_0$. Here $L = B_r(x_0) \cap \{u > l\}$. The symbol γ denotes a constant depending only on p, q, n, μ_1, μ_2, whose meaning can be varied during the paper.

Proof. Let us substitute $\varphi = \left(\int\limits_l^u (1+\frac{s-l}{\delta})^{-1-\lambda}\,ds\right)_+ \xi^k$ as a truncated function in the integral identity (2.3) corresponding to Eq. (2.1). Using conditions 2)–3), Young inequality, $|a(x)-a(x_0)| \le [a]_\alpha r^\alpha$, $\forall\, x \in B_r(x_0)$ and the inequality $\left(\int\limits_l^u (1+\frac{s-l}{\delta})^{-1-\lambda}\,ds\right)_+ \le \gamma\delta$, we get the required estimate (3.1). □

Lemma 3.2. *Let $0<\lambda<1$. Then for every solution u to Eq. (2.1), any $0<l$, δ, $k \ge q$ and function $\xi \in C_0^\infty(B_r(x_0))$ satisfying the conditions*

$$0 \le \xi \le 1,\ \xi(x) \equiv 1,\ \forall\, x \in B_{\frac{r}{2}}(x_0),\ |\nabla\xi| \le \frac{2}{r},$$

the following estimate holds:

$$\int\limits_L \left(1+\frac{u-l}{\delta}\right)^{-1-\lambda} |\nabla u|^q \xi^k\,dx$$
$$\le \gamma \left(\frac{\delta}{r}\right)^q \int\limits_L \left(1+\frac{u-l}{\delta}\right)^{(1+\lambda)(q-1)} \xi^{k-q}\,dx + \gamma\delta \int\limits_{B_r(x_0)} f\,dx, \tag{3.2}$$

if $a(x_0)>0$, $r \le \rho_0$. Here $L = B_r(x_0) \cap \{u>l\}$.

Proof. We test (2.3) by function $\varphi = \left(\int\limits_l^u (1+\frac{s-l}{\delta})^{-1-\lambda}\,ds\right)_+ \xi^k$. Using conditions 2)–3), Young inequality, $\frac{1}{2}a(x_0) \le a(x) \le \frac{3}{2}a(x_0)$, $\forall\, x \in B_r(x_0)$ and the inequality $\left(\int\limits_l^u (1+\frac{s-l}{\delta})^{-1-\lambda}\,ds\right)_+ \le \gamma\delta$, we get the required estimate (3.2). □

3.2. Proof of Theorem 2.2

We will apply here the well-known Kilpeläinen–Maly technique [1] in order to prove the estimates in the Theorem 2.2. First of all, we consider the case $a(x_0)=0$ and introduce the necessary notations.

We set

$$r_j = \frac{\rho}{2^j},\ B_j = B_{r_j}(x_0),\ j=0,\,1,\,2,\ldots,$$

and

$$A_j(l) := r_j^{-n} \int\limits_{B_j \cap \{u>l_j\}} \left(\frac{u-l_j}{l-l_j}\right)^{(1+\lambda)(q-1)} \xi_j^{k-q}\,dx$$
$$+\frac{(l-l_j)^{(q-p)\frac{n}{p}}}{r_j^n} \int\limits_{B_j \cap \{u>l_j\}} \left(\frac{u-l_j}{l-l_j}\right)^{(1+\lambda)(q-1)} \xi_j^{k-q}\,dx.$$

Denote

$$\delta_j(l) = l-l_j,\ L_j = B_j \cap \{u>l_j\},$$

$$\xi_j \in C_0^\infty(B_j),\ 0 \le \xi_j \le 1,\ \xi_j(x) \equiv 1,\ \forall\, x \in B_{j+1},\ |\nabla\xi_j| \le \frac{2}{r_j}.$$

The sequences $\{l_j\}$, $\{\delta_j\}$ will be defined by such a way.

Take $l_0 = 0$ and set

$$\delta_0 = \left(\kappa^{-1} r_0^{-n} \int\limits_{B_j \cap \{u>0\}} u^{(1+\lambda)(q-1)} dx\right)^{\frac{1}{(1+\lambda)(q-1)}} + \left(\kappa^{-1} r_0^{-n} \int\limits_{B_j \cap \{u>0\}} u^{(1+\lambda)(q-1)} dx\right)^{\frac{1}{(1+\lambda)(q-1)-(q-p)\frac{n}{p}}},$$

where $\kappa \in (0, 1)$ will be chosen later.

It is obvious that $A_0(\delta_0) \le \kappa$. We set $l_1 = \delta_0$ and fix k from the equality $2^n(k^{-(1+\lambda)(q-1)} + k^{-(1+\lambda)(q-1)+(q-p)\frac{n}{p}}) = \frac{1}{2}$. Assume that we have already chosen $l_2, \ldots, l_j$ and $\delta_1, \ldots, \delta_{j-1}$ so that $\delta_i = l_{i+1} - l_i$, $i = 1, \ldots, j-1$,

$$l_{i-1} + \frac{1}{2}\delta_{i-2} \le l_i < l_{i-1} + k\delta_{i-2},\ i = 2, \ldots, j, \tag{3.3}$$

$$A_{i-1}(l_i) \le \kappa,\ i = 2, \ldots, j. \tag{3.4}$$

By virtue of the choice of the number k, from the last inequality we have

$$A_j(l_j + k\delta_{j-1}) \le 2^n(k^{-(1+\lambda)(q-1)} + k^{-(1+\lambda)(q-1)+(q-p)\frac{n}{p}}) A_{j-1}(l_j) \le \frac{1}{2}\kappa.$$

Let us clarify now the choice of l_{j+1} and δ_j.

If $A_j(l_j + \frac{1}{2}\delta_{j-1}) \le \kappa$, then we set $l_{j+1} = l_j + \frac{1}{2}\delta_{j-1}$. If $A_j(l_j + \frac{1}{2}\delta_{j-1}) > \kappa$, then there exists $\tilde{l} \in (l_j + \frac{1}{2}\delta_{j-1}, l_j + k\delta_{j-1})$ such that $A_j(\tilde{l}) = \kappa$. In this case we set $l_{j+1} = \tilde{l}$. We used the properties of continuity and decreasing behavior of the function $A_j(l)$.

The following lemma underlies the Kilpeläinen–Maly method [1] and is the basic auxiliary result for the proof of the estimates of Theorem 2.2.

Lemma 3.3. *Let $a(x_0) = 0$. Then for all $j \ge 2$, the following estimate holds:*

$$\delta_j \le \frac{1}{2}\delta_{j-1} + \gamma \left(r_j^{p-n} \int\limits_{B_j} f\, dx\right)^{\frac{1}{p-1}}. \tag{3.5}$$

Proof. Let us fix some $j \ge 1$. Without loss of generality we assume that

$$\delta_j \ge \frac{1}{2}\delta_{j-1}.$$

Otherwise, inequality (3.5) is obvious. Let us establish now that $A_j(l_{j+1}) = \kappa$.

For this purpose we decompose $L_j = L_j' \cup L_j''$. Here $L_j' := \{x \in L_j : \frac{u-l_j}{\delta_j} < \varepsilon\}$. The small parameter $\varepsilon > 0$ will be determined below. Using conditions on q and $\xi_{j-1} \equiv 1$ in B_j, we have

$$r_j^{-n} \int\limits_{L_j'} \left(\frac{u-l_j}{\delta_j}\right)^{(1+\lambda)(q-1)} \xi_j^{k-q}\, dx + r_j^{-n} \delta_j^{(q-p)\frac{n}{p}} \int\limits_{L_j'} \left(\frac{u-l_j}{\delta_j}\right)^{(1+\lambda)(q-1)} \xi_j^{k-q}\, dx$$

$$\le \varepsilon^{(1+\lambda)(q-1)} r_j^{-n} \int\limits_{L_j} \xi_{j-1}^{k-q} dx + \varepsilon^{(1+\lambda)(q-1)-(q-p)\frac{n}{p}} r_j^{-n} \int\limits_{L_j} (u - l_{j-1})^{(q-p)\frac{n}{p}} \xi_{j-1}^{k-q} dx$$

$$\leq \varepsilon^{\lambda(q-1)} r_j^n \int\limits_{L_j} \left(\frac{u-l_{j-1}}{\delta_{j-1}}\right)^{(1+\lambda)(q-1)} \xi_{j-1}^{k-q}\, dx + \varepsilon^{\lambda(q-1)} \delta_{j-1}^{(q-p)\frac{n}{p}} r_j^n \int\limits_{L_j} \left(\frac{u-l_{j-1}}{\delta_{j-1}}\right)^{(1+\lambda)(q-1)} \xi_{j-1}^{k-q}\, dx$$

$$\leq 2^n \varepsilon^{\lambda(q-1)} A_{j-1}(l_j) \leq 2^n \varepsilon^{\lambda(q-1)} \kappa. \tag{3.6}$$

We set $w = \frac{1}{\delta_j}\left(\int\limits_{l_j}^{u} \left(1+\frac{s-l_j}{\delta_j}\right)^{\frac{1+\lambda}{p}} ds\right)_+$ and note that estimate

$$\gamma^{-1}(\varepsilon)\left(\frac{u-l_j}{\delta_j}\right)^{p-1-\lambda} \leq w^p \leq \gamma(\varepsilon)\left(\frac{u-l_j}{\delta_j}\right)^{p-1-\lambda}$$

holds on L_j''. Therefore,

$$r_j^{-n}\left(1+\delta_j^{(q-p)\frac{n}{p}}\right) \int\limits_{L_j''} \left(\frac{u-l_j}{\delta_j}\right)^{(1+\lambda)(q-1)} \xi_j^{k-q}\, dx$$

$$\gamma(\varepsilon) r_j^{-n}\left(1+\delta_j^{(q-p)\frac{n}{p}}\right) \int\limits_{L_j''} w^{\frac{p(1+\lambda)(q-1)}{p-1-\lambda}} \xi_j^{k-q}\, dx.$$

We choose λ from the condition $\frac{p(1+\lambda)(q-1)}{p-1-\lambda} < \frac{n}{n-p}$. So, $\lambda < \frac{p(n-1)-q(n-p)}{n+(q-p)(n-p)}$. With the help of Lemma 3.1, embedding theorem and the previous inequality, we obtain

$$r_j^{-n}\left(1+\delta_j^{(q-p)\frac{n}{p}}\right) \int\limits_{L_j''} \left(\frac{u-l_j}{\delta_j}\right)^{(1+\lambda)(q-1)} \xi_j^{k-q}\, dx$$

$$\leq \gamma(\varepsilon)\left(r_j^{-n} \int\limits_{L_j} \left(1+\frac{u-l_j}{\delta_j}\right)^{(1+\lambda)(q-1)} \xi_j^{(k-q)\frac{n-p}{n}-q}\, dx \right.$$

$$\left. + r_j^{-n}\delta_j^{q-p} \int\limits_{L_j} \left(1+\frac{u-l_j}{\delta_j}\right)^{(1+\lambda)(q-1)} \xi_j^{(k-q)\frac{n-p}{n}-q}\, dx + r_j^{p-n}\delta_j^{1-p} \int\limits_{B_j} f\, dx \right)^{\frac{n}{n-p}}$$

$$+\gamma(\varepsilon)\left(r_j^{-n}\delta_j^{(q-p)\frac{n-p}{p}} \int\limits_{L_j} \left(1+\frac{u-l_j}{\delta_j}\right)^{(1+\lambda)(q-1)} \xi_j^{(k-q)\frac{n-p}{n}-q}\, dx \right.$$

$$\left. + r_j^{-n}\delta_j^{(q-p)\frac{n}{p}} \int\limits_{L_j} \left(1+\frac{u-l_j}{\delta_j}\right)^{(1+\lambda)(q-1)} \xi_j^{(k-q)\frac{n-p}{n}-q}\, dx + r_j^{p-n}\delta_j^{1-p+(q-p)\frac{n}{p}} \int\limits_{B_j} f\, dx \right)^{\frac{n}{n-p}}.$$

Due to Young inequality, choice of k from the condition $(k-q)\frac{n-p}{n} - q = 1$ and the estimate

$$r_j^{-n}\delta_j^{(q-p)\frac{n}{p}} \int\limits_{L_j} \left(1+\frac{u-l_j}{\delta_j}\right)^{(1+\lambda)(q-1)} \xi_j\, dx \leq \gamma\kappa,$$

we have

$$r_j^{-n}(1+\delta_j^{(q-p)\frac{n}{p}}) \int\limits_{L_j''} \left(\frac{u-l_j}{\delta_j}\right)^{(1+\lambda)(q-1)} \xi_j^{k-q}\, dx$$

$$\leq \gamma(\varepsilon) \left(\kappa + \delta_j^{1-n+q\frac{n-p}{p}} r_j^{p-n} \int\limits_{B_j} f\, dx\right)^{\frac{n}{n-p}}. \tag{3.7}$$

So, it is clearly to see that estimates (3.6) and (3.7) yield the inequality

$$\kappa \leq 2^n \varepsilon^{\lambda(p-1)} \kappa + \gamma(\varepsilon) \left(\kappa + \delta_j^{1-p} r_j^{p-n} \int\limits_{B_j} f\, dx\right)^{\frac{n}{n-p}}$$

$$+\gamma(\varepsilon) \left(\kappa + \delta_j^{1-p+(q-p)\frac{n}{p}} r_j^{p-n} \int\limits_{B_j} f\, dx\right)^{\frac{n}{n-p}}. \tag{3.8}$$

We choose ε to be sufficiently small, $2^n \varepsilon^{\lambda(p-1)} = \frac{1}{4}$. Let us choose $\kappa = \kappa(\varepsilon)$ as follows $\gamma(\varepsilon)\kappa^{\frac{p}{n-p}} = \frac{1}{4}$. Estimate (3.8) implies that at least one of the inequalities

$$\delta_j \leq \gamma \left(r_j^{p-n} \int\limits_{B_j} f\, dx\right)^{\frac{1}{p-1}}$$

or

$$\delta_j \leq \gamma \left(r_j^{p-n} \int\limits_{B_j} f\, dx\right)^{\frac{1}{n-1-q\frac{n-p}{p}}}$$

holds. From assumption of series convergence $\sum\limits_{j=0}^{\infty} \left(r_j^{p-n} \int\limits_{B_j} f\, dx\right)^{\frac{1}{p-1}}$, we get the required estimate (3.5). The lemma is proved. □

Analogously, we can establish:

Remark 3.1. Under the conditions $a(x_0) > 0$ and $\rho_0 \geq \rho$ the estimate

$$\delta_j \leq \frac{1}{2}\delta_{j-1} + r_j + \gamma \left(r_j^{q-n} \int\limits_{B_j} f\, dx\right)^{\frac{1}{q-1}} \tag{3.9}$$

is true for all $j \geq 2$.

For $a(x_0) > 0$ and $\rho_0 < \rho$ there exists $j_0 > 1 : \frac{\rho}{2^{j_0+1}} < \rho \leq \frac{\rho_0}{2^{j_0}}$, such that

$$\delta_j \leq \frac{1}{2}\delta_{j-1} + r_j + \gamma \left(r_j^{p-n} \int\limits_{B_j} f\, dx\right)^{\frac{1}{p-1}}, \tag{3.10}$$

for all $1 \le j < j_0$.

In the case $j \ge j_0$ estimate (3.9) is valid:

$$\delta_j \le \frac{1}{2}\delta_{j-1} + r_j + \gamma \left(r_j^{q-n} \int\limits_{B_j} f\, dx \right)^{\frac{1}{q-1}}. \tag{3.9}$$

Using proved auxiliary assertions (Lemmas 3.1–3.3), Remark 3.1, we complete the proof of Theorem 2.2.

Let us sum inequalities (3.5) for $j = 2, \ldots, J$:

$$l_J \le l_1 + \gamma\delta_1 + \sum_{j=0}^{\infty} \left(r_j^{p-n} \int\limits_{B_j} f\, dx \right)^{\frac{1}{p-1}}.$$

Since $l_1 = \delta_0$ and $\delta_1 \le k\delta_0$, we have

$$l_J \le \gamma\delta_0 + \sum_{j=0}^{\infty} \left(r_j^{p-n} \int\limits_{B_j} f\, dx \right)^{\frac{1}{p-1}},$$

where δ_0 has been already defined above.

In the last inequality we pass to the limit $J \to \infty$. Let $l := \lim\limits_{j\to\infty} l_j$. Then we obtain

$$u(x_0) \le l \le \gamma\delta_0 + \sum_{j=0}^{\infty} \left(r_j^{p-n} \int\limits_{B_j} f\, dx \right)^{\frac{1}{p-1}}.$$

Here x_0 is the Lebesgue point of the function $(u-l)_+^{(1+\lambda)(p-1)}$. Due to definition of δ_0, we arrive to estimate (2.8). To prove estimate (2.9), we sum inequalities (3.9) for$j = 2, \ldots, J$. As a result, we have

$$l_J \le \gamma\delta_0 + 2\rho + \gamma W_{1,q}^{f}(x_0, 2\rho). \tag{3.11}$$

The definition of l_1 yields $\delta_0 < \infty$. So, the sequence $\{l_j\}_{j\in\mathbb{N}}$ is convergent and $\delta_j \to 0$ as $j \to \infty$. Passing to the limit $J \to \infty$ in (3.11) and setting $l := \lim\limits_{j\to\infty} l_j$, we obtain:

$$\frac{1}{r_j^n} \int\limits_{B_j} (u-l)_+^{(1+\lambda)(q-1)} \le \gamma\delta_j^{(1+\lambda)(q-1)} \to 0,\ j \to \infty.$$

Let x_0 be chosen as the Lebesgue point of the function $(u-l)_+^{(1+\lambda)(q-1)}$. Then we get $u(x_0) \le l \le \gamma\delta_0 + 2\rho + \gamma W_{1,q}^{f}(x_0, 2\rho)$. Thus, estimate (2.9) is also proved.

We consider now the case $a(x_0) > 0$, $\rho_0 < \rho$ and prove estimate (2.10). For this purpose, we use Remark 3.1 and sum (3.10) for $j = 2, \ldots, j_0 - 1$ and (3.9) for $j = j_0, j_0+1, \ldots, J$.

As a result, we get

$$l_J \le \gamma\delta_0 + 2\rho + \gamma(W_{1,q}^{f}(x_0, 2\rho_0) + (W_{1,p}^{f}(x_0, 2\rho) - W_{1,p}^{f}(x_0, 2\rho_0))). \tag{3.12}$$

The definition of l_1 yields $\delta_0 < \infty$. So, the sequence $\{l_j\}_{j\in\mathbb{N}}$ is convergent and $\delta_j \to 0$ as $j \to \infty$. Let us pass to the limit $J \to \infty$ in (3.12) and let $l := \lim\limits_{j\to\infty} l_j$. Then

$$\frac{1}{r_j^n} \int\limits_{B_j} (u-l)_+^{(1+\lambda_0)(p-1)} \le \gamma \delta_j^{(1+\lambda_0)(p-1)} \to 0,\ j \to \infty.$$

Choosing x_0 as the Lebesgue point of the function $(u-l)_+^{(1+\lambda_0)(p-1)}$, we have $u(x_0) \le l$.

Thus, estimate (2.10) and Theorem 2.2 are completely proved.

4. Proof of Theorem 2.1

The right estimates of Theorem 2.1 will be consequence of Theorem 2.2 and the weak Harnack inequality obtained earlier in [17] for double-phase functionals

$$\left(\int\limits_{B_\rho(x_0)} u^s \, ds \right)^{\frac{1}{s}} \le \inf_{x\in B_\rho(x_0)} u, \tag{4.1}$$

with some exponent $s > 0$. Indeed, from the inequality (4.1), Theorem 2.2 and estimate (2.8), we arrive to

$$u(x_0) \le c_6 \inf_{B_\rho(x_0)} u + c_6 W^f_{1,p}(x_0, 2\rho),$$

if $a(x_0) = 0$.

If $a(x_0) > 0$ and $\rho_0^\alpha = \frac{a(x_0)}{4[a]_{C^{0,\alpha}(\Omega)}} \ge \rho^\alpha$, estimates (4.1) and (2.9) yield

$$u(x_0) \le 3\rho + c_8 \inf_{B_\rho(x_0)} u + c_8 W^f_{1,q}(x_0, 2\rho).$$

Moreover, (4.1) and (2.10) yield the estimate

$$u(x_0) \le 3\rho_0 + c_8 \inf_{B_\rho(x_0)} u + c_8 W^f_{1,q}(x_0, 2\rho_0) + c_8 (W^f_{1,p}(x_0, 2\rho) - W^f_{1,p}(x_0, 2\rho_0)).$$

It remains to prove the lower bounds in (2.4)–(2.6). For this purpose, we test (2.3) by $\varphi = \xi^q$, $\xi \in C_0^\infty(B_r(x_0))$, $0 \le \xi \le 1$, $\xi \equiv 1$ in $B_{\frac{r}{2}}(x_0)$ and $|\nabla \xi| \le \frac{\gamma}{r}$, $0 < r \le \rho$. Let us note that

$$g(a(x), a)b \le \varepsilon g(a(x), a)a + g\left(a(x), \frac{b}{\varepsilon}\right) b,\ a,\ b,\ \varepsilon > 0. \tag{4.2}$$

In addition, it is clearly to see $\frac{3}{4}a(x_0) \le a(x) \le \frac{5}{4}a(x_0)$, $\forall x \in B_\rho(x_0)$. So, we get

$$\left(\frac{3}{4}\right)^{q-1} g(a(x_0),\, t) \le g(a(x),\, t) \le \left(\frac{5}{4}\right)^{p-1} g(a(x_0),\, t), \tag{4.3}$$

if $a(x_0) > 0$ and $\rho_0 \ge \rho$.

From conditions 2)–3), estimates (4.2) and (4.3) with $\varepsilon = g^{\beta}\left(a(x_0), \frac{m\left(\frac{r}{2}\right)-m(r)}{r}\right)$, $0 < \beta < \min\left(1, \frac{1}{(n-1)(q-1)}\right)$, $m(r) = \inf\limits_{B_r(x_0)} u$, we have

$$\int\limits_{B_{\frac{r}{2}}(x_0)} f\,dx \le \gamma \int\limits_{B_r(x_0)} g(a(x_0), |\nabla u|)|\nabla\xi|\,\xi^{q-1}dx$$

$$\le \gamma\varepsilon \int\limits_{B_r(x_0)} \psi^{-\beta}\left(\frac{u-m(r)}{r}\right)\frac{G(a(x_0), |\nabla u|)}{u-m(r)}\xi^q\,dx$$

$$+\frac{\gamma}{r}\int\limits_{B_r(x_0)} g\left(a(x_0), \frac{1}{\varepsilon}\frac{u-m(r)}{r}\psi^{\beta}\left(\frac{u-m(r)}{r}\right)\right)dx. \tag{4.4}$$

Let us substitute $\varphi = \psi^{-\beta}\left(\frac{u}{r}\right)\xi^q$ as φ in (2.3). Using conditions (2.2) and the weak Harnack inequality, we obtain

$$\varepsilon \int\limits_{B_r(x_0)} \psi^{-\beta}\left(\frac{u-m(r)}{r}\right)\frac{G(a(x_0), |\nabla u|)}{u-m(r)}\xi^q\,dx$$

$$\le \gamma r^{-1}\varepsilon \int\limits_{B_r(x_0)} \psi^{1-\beta}\left(\frac{u-m(r)}{r}\right)dx$$

$$\le \gamma r^{-1}\varepsilon \int\limits_{B_r(x_0)} g^{1-\beta}\left(a(x_0), \frac{u-m(r)}{r}\right)dx$$

$$\le \gamma r^{n-1} g\left(a(x_0), \frac{m\left(\frac{r}{2}\right)-m(r)}{r}\right). \tag{4.5}$$

Since $0 < \beta < \min\left(1, \frac{1}{(n-1)(q-1)}\right)$, we have

$$\gamma r^{-1}\int\limits_{B_r(x_0)} g\left(a(x_0), \frac{1}{\varepsilon}\frac{u-m(r)}{r}g^{\beta}\left(a(x_0), \frac{u-m(r)}{r}\right)\right)dx$$

$$\le \gamma r^{n-1} g\left(a(x_0), \left(\frac{m\left(\frac{r}{2}\right)-m(r)}{r}\right)\right)$$

$$+\gamma r^{-1}\varepsilon^{1-q}\int\limits_{B_r(x_0)} g^{1+\beta(q-1)}\left(a(x_0), \frac{u-m(r)}{r}\right)dx$$

$$\le \gamma r^{n-1} g\left(a(x_0), \left(\frac{m\left(\frac{r}{2}\right)-m(r)}{r}\right)\right). \tag{4.6}$$

Relations (4.4)–(4.6) yield

$$r^{1-n} \int\limits_{B_{\frac{r}{2}}(x_0)} f\,dx \le g\left(a(x_0), \frac{m\left(\frac{r}{2}\right) - m(r)}{r}\right). \tag{4.7}$$

We note that for $a(x_0) > 0$ the following inequality is true:

$$g\left(a(x_0), \frac{m\left(\frac{r}{2}\right) - m(r)}{r}\right) \le 1 + (\gamma + a(x_0)) \left(\frac{m\left(\frac{r}{2}\right) - m(r)}{r}\right)^{q-1}. \tag{4.8}$$

If $a(x_0) = 0$, then

$$g\left(0, \frac{m\left(\frac{r}{2}\right) - m(r)}{r}\right) \le 1 + \gamma \left(\frac{m\left(\frac{r}{2}\right) - m(r)}{r}\right)^{p-1}.$$

By integrating inequality (4.7) over $r \in (0,\, \rho)$ and using the previous estimates, we get the lower bounds in (2.4) and (2.5).

To establish the lower bound in (2.6), we use the proved estimates (4.7) and (4.8).

Integrating inequality (4.7) over $r \in (0,\, \rho)$, we have to necessity to divide the interval of integration into $r \in (0,\, \rho_0)$ and $r \in (\rho_0,\, \rho)$.

Since

$$\int\limits_0^{\rho_0} \left(r^{q-n} \int\limits_{B_{\frac{r}{2}}(x_0)} f\,dx \right)^{\frac{1}{q-1}} dr = W_{1,q}^{f}(x_0, \rho_0)$$

and

$$\int\limits_{\rho_0}^{\rho} \left(r^{p-n} \int\limits_{B_{\frac{r}{2}}(x_0)} f\,dx \right)^{\frac{1}{p-1}} dr$$

$$= \int\limits_0^{\rho} \left(r^{p-n} \int\limits_{B_{\frac{r}{2}}(x_0)} f\,dx \right)^{\frac{1}{p-1}} dr - \int\limits_0^{\rho_0} \left(r^{p-n} \int\limits_{B_{\frac{r}{2}}(x_0)} f\,dx \right)^{\frac{1}{p-1}} dr$$

$$= W_{1,p}^{f}(x_0, \rho) - W_{1,p}^{f}(x_0, \rho_0),$$

we get the lower bound in (2.6).

Theorem 2.1 is completely proved.

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