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Keller–Osserman a priori estimates and the removability result for the anisotropic porous medium equation with absorption term

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Abstract. We obtain the removability result for quasilinear equations of the form

$$u_t - \sum_{i=1}^{n} \left(u^{m_i-1} u_{x_i}\right)_{x_i} + f(u) = 0,\ u \geq 0,$$

and prove a priori estimates of the Keller–Osserman type.

Keywords. Anisotropic porous medium equation, Keller–Osserman a priori estimates, removability of an isolated singularity.

1. Introduction and main results

We will study solutions to a quasilinear parabolic equation in the divergent form

$$u_t - divA(x,t,u,\nabla u) + a_0(u) = 0,\ (x,t) \in \Omega_T = \Omega \times (0,T), \tag{1.1}$$

satisfying the initial condition

$$u(x,0) = 0,\ \ x \in \Omega \setminus \{0\}, \tag{1.2}$$

where Ω is a bounded domain in $R^n, n \geq 2, 0 < T < \infty$.

We suppose that the functions $A = (a_1, \ldots, a_n)$ and a_0 satisfy the Carathéodory conditions, and the following structure conditions hold:

$$A(x,t,u,\xi)\xi \geq \nu_1 \sum_{i=1}^{n} |u|^{m_i-1}|\xi_i|^2,$$

$$|a_i(x,t,u,\xi)| \leq \nu_2 u^{(m_i-1)\frac{1}{2}} \left(\sum_{j=1}^{n} |u|^{m_j-1}|\xi_j|^2\right)^{\frac{1}{2}},\ i = \overline{1,n}, \tag{1.3}$$

$$a_0(u) \geq \nu_1 f(u),$$

with positive constants ν_1 and ν_2 and a continuous positive function $f(u)$. Moreover,

$$\min_{1\leq i\leq n} m_i > 1 - \frac{2}{n},\ \max_{1\leq i\leq n} m_i \leq m + \frac{2}{n}, \tag{1.4}$$

where $m = \frac{1}{n}\sum_{i=1}^{n} m_i$. Without loss of generality, we assume that $m_1 \leq m_2 \leq \ldots \leq m_n$.

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Many authors studied the problems of singularities of the solutions of second-order quasilinear elliptic and parabolic equations. The review of these results can be found in [19]. Brezis and Veron [2] proved that the isolated singularities of solutions to the elliptic equation

$$-\triangle u + u^q = 0,$$

are removable for $q \geq n/(n-2)$. In [3], Brézis and Friedman proved that the isolated singularities of solutions for the parabolic equation

$$\frac{\partial u}{\partial t} - \triangle u + |u|^{q-1}u = 0, \ \ (x,t) \in \Omega_T \setminus \{(0,0)\}$$

are removable for $q \geq (n+2)/n$. The removability of an isolated singularity for solutions of porous medium equation in the nonanisotropic case $(m = m_1 = \ldots = m_n)$

$$u_t - \triangle\left(|u|^{m-1}u\right) + |u|^{q-1}u = 0,$$

has been proved under the assumption $q \geq m + \frac{2}{n}$ by Kamin and Peletier [5].

The development of the qualitative theory of second-order quasilinear elliptic and parabolic equations with nonstandard growth conditions has been observed in recent decades. It is worth to mention works [4, 6, 7, 9, 10, 12, 15–18]. One of the basic prototypes of such equations is

$$\frac{\partial u}{\partial t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(|u|^{(m_i-1)(p_i-2)} \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i}\right) = 0, p_i \geq 2, m_i \geq 1, i = \overline{1,n}.$$

The removability result of an isolated singularity and a priori estimates of the Keller–Osserman type for this equation were obtained in [9, 12].

We now define a weak solution to problem (1.1), (1.2) with the singularity at the point $(0,0)$. We write $V_{2,m}(\Omega_T)$ for the class of functions $\varphi \in C_{loc}(0,T, L_{loc}^{1+m^-}(\Omega))$ with $\sum\limits_{i=1}^{n} \iint\limits_{\Omega_T} |\varphi|^{m_i+m^- -2} |\varphi_{x_i}|^2 \, dxdt < \infty$, where $m^- = \min(m_n, 1)$. By a weak solution to problem (1.1), (1.2) we mean a function $u(x,t) \geq 0$ satisfying the inclusion $u\psi \in V_{2,m}(\Omega_T) \cap L^2_{loc}(0,T; W^{1,2}_{loc}(\Omega))$. The integral identity

$$\int\limits_{\Omega} u(x,\tau)\psi\varphi dx + \int\limits_{0}^{\tau}\int\limits_{\Omega} \{-u(\psi\varphi)_t + A(x,t,u,\nabla u)\nabla(\psi\varphi) + a_0(u)\psi\varphi\} \, dx\, dt = 0 \qquad (1.5)$$

holds for any testing function $\varphi \in W^{1,2}_{loc}(0,T; L^2_{loc}(\Omega)) \cap L^2_{loc}(0,T; \overset{o}{W}{}^{1,2}_{loc}(\Omega))$, any $\psi \in C^1(\overline{\Omega}_T)$ vanishing in the neighborhood of $\{(0,0)\}$, and for all $\tau \in (0,T)$.

The result of this paper is the removability of isolated singularities for solutions to the anisotropic porous medium equation with absorption term. The proof of this result is based on a priori estimates of the Keller–Osserman type of the solution to Eq. (1.1). The main difficulty lies in the fact that a part of $m_i < 1$ (singular case), and another part of $m_i > 1$ (degenerate case).

Theorem 1.1. *Let conditions* (1.3) *and* (1.4) *be fulfilled, and let u be a nonnegative weak solution to the problem* (1.1), (1.2). *Assume also that $f(u) = u^q$ and*

$$q \geq m + \frac{2}{n}. \qquad (1.6)$$

Then the singularity at the point $\{(0,0)\}$ is removable.

Let $(x^{(0)},t^{(0)}) \in \Omega_T$. For any $\tau,\theta_1,\theta_2,\dots,\theta_n > 0$, $\theta=(\theta_1,\dots,\theta_n)$, we define $Q_{\theta,\tau}(x^{(0)},t^{(0)}) := \{(x,t) : |t-t^{(0)}|<\tau,\ |x_i-x_i^{(0)}|<\theta_i,\ i=\overline{1,n}\}$ and set $M(\theta,\tau) := \sup\limits_{Q_{\theta,\tau}(x^{(0)},t^{(0)})} u$, $F(\theta,\tau) := \sup\limits_{Q_{\theta,\tau}(x^{(0)},t^{(0)})} F(u)$, $F(u)=\int\limits_0^u s^{m^- -1}\, f(s)ds$, $m^+=\max(m_n,1)$.

Theorem 1.2. *Let conditions* (1.3) *and* (1.4) *be fulfilled, and let u be a nonnegative weak solution to Eq.* (1.1). *Assume also that $f \in C^1(R^1_+)$ and $f'(u) \geq 0$. Let $(x^{(0)},t^{(0)}) \in \Omega_T$, fix $\sigma\in(0,1)$, and let $Q_{8\theta,8\tau}(x^{(0)},t^{(0)}) \subset \Omega_T$. Set $\rho = \begin{cases} \theta_n, \text{if } m_n>1, \\ \tau^{\frac12}, \text{if } m_n<1, \end{cases}$ Then there exist positive numbers c_1 and c_2 depending only on $n,\nu_1,\nu_2,m_1,\dots,m_n$ such that either*

$$u(x^{(0)},t^{(0)}) \leq \left(\frac{\theta_n^2}{\tau}\right)^{\frac{1}{m_n-1}} + \sum_{i=1}^{n-1}\left(\frac{\rho}{\theta_i}\right)^{\frac{2}{m^+-m_i}} \tag{1.7}$$

or

$$(M(\sigma\theta,\sigma\tau))^{1-m^-+\frac{n(m-m^-)}{2}} F(M(\sigma\theta,\sigma\tau)) \leq c_1(1-\sigma)^{-\gamma}\rho^{-2}(M(\theta,\tau))^{m^++1+\frac{n(m-m^-)}{2}} \tag{1.8}$$

holds true.

In particular, if

$$F(\varepsilon u) \leq \varepsilon^{m^++m^-+\beta} F(u),\ \beta>0, \tag{1.9}$$

then

$$F(M(\theta,\tau)) \leq c_2(1-\sigma)^{-\gamma} M^{m^++m^-}(\theta,\tau)\rho^{-2}. \tag{1.10}$$

An example of the function f, which satisfies condition (1.9), is $f(u)=u^q, q \geq m+\frac{2}{n}$. Assuming for simplicity that $dist(x^{(0)},\partial\Omega)=|x^{(0)}|$ and choosing $\tau,\ \theta_i,$ from the conditions

- $m_n>1$: $\left(\frac{\theta_n^2}{\tau}\right)^{\frac{1}{m_n-1}} = \theta_n^{-\frac{2}{q-m_n}}$, i.e., $\tau=\theta_n^{\frac{2(q-1)}{q-m_n}}$,

  $\left(\frac{\theta_n}{\theta_i}\right)^{\frac{2}{m_n-m_i}} = \theta_n^{-\frac{2}{q-m_n}}$, i.e. $\theta_i=\theta_n^{\frac{q-m_i}{q-m_n}}$,

- $m_n<1$: $\left(\frac{\theta_n^2}{\tau}\right)^{\frac{1}{m_n-1}} = \tau^{-\frac{1}{q-m_n}}$, i.e., $\tau=\theta_n^{\frac{2(q-1)}{q-m_n}}$,

  $\left(\frac{\tau^{\frac12}}{\theta_i}\right)^{\frac{2}{1-m_i}} = \tau^{-\frac{1}{q-1}}$, i.e., $\theta_i=\tau^{\frac{q-m_i}{2(q-1)}}$,

we obtain an estimate from (1.7), (1.10)

$$u(x^{(0)},t^{(0)}) \leq c\left(\sum_{i=1}^n |x_i^{(0)}|^{\frac{2}{q-m_i}} + (t^{(0)})^{\frac{1}{q-1}}\right)^{-1}. \tag{1.11}$$

2. Keller–Osserman a priori estimates

2.1. Auxiliary propositions

Let $E(2\rho)=\{(x,t)\in\Omega_T : u(x,t)>M(2\rho)\}$, $u^{(\rho)}(x,t)=\min\left(M\left(\frac{\rho}{2}\right)-\ M(2\rho),\, u(x,t)-M(2\rho)\right)$.

Lemma 2.1. [11] *Under the assumptions of Theorem 1.1, the following inequality holds:*

$$\iint\limits_{E(2\rho)} u^{(\rho)} u^q \psi_r^l \, dxdt \le \gamma \left(M\left(\frac{\rho}{2}\right) - M(2\rho) \right)$$

$$\times \left\{ F_3(r,\lambda) + (F_1(r,\lambda) + F_2(r,\lambda))^{\frac{1}{2}} F_4^{\frac{1}{2}}(r,\lambda) \right\}, \tag{2.1}$$

where

$$F_1(r,\lambda) = \begin{cases} R^{\lambda}(r), \ \lambda > 0, \\ \ln^{\frac{q-2}{q-1}} \dfrac{1}{r}, \ \lambda = 0, \ \ q > 2, \\ \ln\ln \dfrac{1}{r}, \ \lambda = 0, \ \ q = 2, \\ \ln^{-\frac{2-q}{q-1}} \dfrac{1}{r}, \ \lambda = 0, \ \ q < 2, \end{cases}$$

$$F_2(r,\lambda) = \begin{cases} R^{\lambda}(r), \ \lambda > 0, \\ \ln^{\frac{q-2m_1}{q-m_1}} \dfrac{1}{r}, \ \lambda = 0, \ \ q > 2m_1, \\ \ln\ln \dfrac{1}{r}, \ \lambda = 0, \ \ q = 2m_1, \\ \ln^{-\frac{2m_1-q}{1-m_1}} \dfrac{1}{r}, \ \lambda = 0, \ \ q < 2m_1, \end{cases}$$

$$F_3(r,\lambda) = \begin{cases} R^{\lambda}(r), \ \lambda > 0, \\ \ln^{-\frac{1}{q-1}} \dfrac{1}{r}, \ \lambda = 0, \end{cases} \quad F_4(r,\lambda) = \begin{cases} R^{\lambda}(r), \ \lambda > 0, \\ \ln^{-1} \dfrac{1}{R(r)}, \ \lambda = 0, \end{cases}$$

where $\lambda = n - \frac{2}{q-m}, 0 < r < R_0$.

Lemma 2.2. [1] *Let* $\Omega \subset R^n$, $n \ge 2$ *be a bounded domain,* $v \in \overset{o}{W}{}^{1,1}(\Omega)$, *and*

$$\sum_{i=1}^{n} \int\limits_{\Omega} |v|^{\alpha_i} |v_{x_i}|^{p_i} dx < \infty, \ \alpha_i \ge 0, \ p_i > 1. \tag{2.2}$$

If $1 < p < n$, *then* $v \in L^q(\Omega), q = \frac{np}{n-p}\left(1 + \frac{1}{n}\sum\limits_{i=1}^{n}\frac{\alpha_i}{p_i}\right)$, $\frac{1}{p} = \frac{1}{n}\sum\limits_{i=1}^{n}\frac{1}{p_i}$, *and the following inequality holds:*

$$\|v\|_{L^q(\Omega)} \le \gamma \prod_{i=1}^{n} \left(\int\limits_{\Omega} |v|^{\alpha_i} |v_{x_i}|^{p_i} dx \right)^{\frac{1}{np_i\left(1+\frac{1}{n}\sum\limits_{i=1}^{n}\frac{\alpha_i}{p_i}\right)}}, \tag{2.3}$$

where the positive constant γ *depends only on* $n, p_i, \alpha_i, i = \overline{1,n}$.

Lemma 2.3. [8, chap. 2] *Let* $\{y_j\}_{j\in N}$ *be a sequence of nonnegative numbers such that for any* $j = 0, 1, 2, \ldots$ *the inequality*

$$y_{j+1} \le C b^j y_j^{1+\varepsilon}$$

holds with positive $\varepsilon, C > 0, b > 1$. *Then the following estimate is true:*

$$y_j \le C^{\frac{(1+\varepsilon)^j - 1}{\varepsilon}} b^{\frac{(1+\varepsilon)^j - 1}{\varepsilon^2} - \frac{j}{\varepsilon}} y_0^{(1+\varepsilon)^j}.$$

Particulary, if $y_0 \le C^{-\frac{1}{\varepsilon}} b^{-\frac{1}{\varepsilon^2}}$, *then* $\lim\limits_{j\to\infty} y_j = 0$.

2.2. Integral estimates of solutions

Consider a cylinder $Q_{\theta,\tau}(x^{(0)},t^{(0)})$. Let $(\bar{x},\bar{t})$ be an arbitrary point in $Q_{\sigma\theta,\sigma\tau}(x^{(0)},t^{(0)})$. If $u(x^{(0)},t^{(0)})\geq \left(\frac{\theta_n^2}{\tau}\right)^{\frac{1}{m_n-1}}+\sum\limits_{i=1}^{n-1}\left(\frac{\rho}{\theta_i}\right)^{\frac{2}{m^+-m_i}}$, then $M(\theta,\tau)=\max(M(\theta,\tau),\delta(\theta,\tau))\geq(\tau^{-1}\theta_n)^{\frac{1}{m_n-1}}+\sum\limits_{i=1}^{n}(\theta_i^{-1}\rho)^{\frac{2}{m^+-m_i}}$. Hence, $Q_{\eta,s}(\bar{x},\bar{t})\subset Q_{\theta,\tau}(x^{(0)},t^{(0)})$, where $s=(1-\sigma)M^{1-m^+}(\theta,\tau)\rho^2$, $\eta_i=(1-\sigma)M^{\frac{m_i-m^+}{2}}(\theta,\tau)\rho$, $i=\overline{1,n}$. For fixed $k>0$ and $l,j=0,1,2\ldots$, set $\alpha_l=\frac{1}{4}(1+2^{-1}+\cdots+2^{-l})$, $k_j=k(1-2^{-j})$, $\eta_{i,j,l}=(\alpha_l+\frac{1}{4}2^{-j-l-1})\eta_i$, $i=\overline{1,n}$, $\eta_{j,l}=(\eta_{1,j,l},\ldots,\eta_{n,j,l})$, $s_{j,l}=(\alpha_l+\frac{1}{4}2^{-j-l-1})s$, $Q_{j,l}=Q_{\eta_{j,l},s_{j,l}}(\bar{x},\bar{t})$, $A_{k_j,j,l}=\{x\in Q_{j,l}(\bar{x},\bar{t}):F(u)>k_j\}$. Let $\xi_j\in C_0^\infty(Q_{j,l}(\bar{x},\bar{t}))$, $0\leq\xi_j\leq1,\xi_j=1$ in $Q_{j+1,l}(\bar{x},\bar{t})$, $\left|\frac{\partial\xi_j}{\partial t}\right|\leq\gamma2^{j+l}s^{-1}$, $\left|\frac{\partial\xi_j}{\partial x_i}\right|\leq\gamma2^{j+l}\eta_i^{-1}, i=\overline{1,n}$.

In what follows, γ stands for a constant depending only on n, ν_1, ν_2, $m_1,\ldots,m_n$ which may vary from line to line.

Lemma 2.4. *Let u be a nonnegative weak solution to Eq.* (1.1), *and let conditions* (1.3) *and* (1.4) *hold. Then for any $j\geq0$ the following inequality holds:*

$$l_j^{1-m^-}\int\limits_{A_{k_j,j,l}(t)}(F(u)-k_j)_+^2\xi_j^2dx+\sum_{i=1}^{n}l_j^{m_i-m^-}\iint\limits_{A_{k_j,j,l}}|\nabla((F(u)-k_j)_+)|^2\,\xi_j^2dxdt$$

$$+\iint\limits_{A_{k_j,j,l}}(F(u)-k_j)_+f^2(u)\xi_j^2dxdt\leq\gamma M^{m^+-m^-}(\theta,\tau)\rho^{-2}\iint\limits_{A_{k_j,j,l}}(F(u)-k_j)_+^2dxdt, \tag{2.4}$$

where $l_j=F^{-1}(k_j), j=0,1,2,\ldots$.

Proof. Testing identity (1.5) by $\varphi=(F(u)-k_j)_+f(u)\xi_j^2$ and using conditions (1.3), we get

$$\iint\limits_{A_{k_j,j,l}}u_tf(u)(F(u)-k_j)_+\xi_j^2dxdt$$

$$+\sum_{i=1}^{n}\iint\limits_{A_{k_j,j,l}}u^{m_i+m^--2}|u_{x_i}|^2f^2(u)\xi_j^2dxdt+\iint\limits_{A_{k_j,j,l}}(F(u)-k_j)_+f^2(u)\xi_j^2dxdt$$

$$\leq\gamma\sum_{i=1}^{n}\iint\limits_{A_{k_j,j,l}}u^{\frac{m_i-1}{2}}\left(\sum_{l=1}^{n}u^{m_l-1}|u_{x_l}|^2\right)^{\frac{1}{2}}(F(u)-k_j)_+f(u)\xi_j\left|\frac{\partial\xi_j}{\partial x_i}\right|dxdt.$$

From this, using the Young inequality and the evident inequality $l_j<u(x,t)<M(\theta,\tau)$ on $A_{k_j,j,l}$, we arrive at the required estimate (2.4). □

2.3. Proof of Theorem 1.2

By Lemma 2.2 and the Hölder inequality, we obtain

$$Y_{j+1,l}=\iint\limits_{A_{k_{j+1},j+1,l}}(F(u)-k_{j+1})_+^2dxdt$$

$$\leq \left|A_{k_{j+1},j+1,l}\right|^{\frac{2}{n+2}} \left(\iint\limits_{A_{k_{j+1},j+1,l}} ((F(u)-k_{j+1})_+\xi_j)^{\frac{2(n+2)}{n}} dxdt \right)^{\frac{n}{n+2}}$$

$$\leq \left|A_{k_{j+1},j+1,l}\right|^{\frac{2}{n+2}} \operatorname*{ess\,sup}_{0<t<T} \left(\int\limits_{A_{k_{j+1},j+1,l}(t)} (F(u)-k_{j+1})_+^2 \xi_j^2 dx \right)^{\frac{2}{n+2}}$$

$$\times \left(\int_0^T \prod_{i=1}^n \left(\int\limits_{A_{k_{j+1},j+1,l}(t)} \left|((F(u)-k_{j+1})_+\xi_j)_{x_i}\right|^2 dx \right)^{\frac{1}{n}} dt \right)^{\frac{n}{n+2}}$$

$$\leq \left|A_{k_{j+1},j+1,l}\right|^{\frac{2}{n+2}} \operatorname*{ess\,sup}_{0<t<T} \left(\int\limits_{A_{k_{j+1},j+1,l}(t)} (F(u)-k_{j+1})_+^2 \xi_j^2 dx \right)^{\frac{2}{n+2}}$$

$$\times \left(\int_0^T \prod_{i=1}^n \left(\int\limits_{A_{k_{j+1},j+1,l}(t)} \left|((F(u)-k_{j+1})_+)_{x_i}\right|^2 \xi_j^2 dx \right.\right.$$

$$\left.\left. + \int\limits_{A_{k_{j+1},j+1,l}(t)} (F(u)-k_{j+1})_+^2 \left|\frac{\partial \xi_j}{\partial x_i}\right|^2 dx \right)^{\frac{1}{n}} dt \right)^{\frac{n}{n+2}}.$$

Denote $Q_l = Q_{\alpha_l\eta,\alpha_l s}, M_l = \sup\limits_{Q_l} u$. In view of (2.4), it follows from Lemma 2.3 that $y_{j,l} \to 0$ as $j \to \infty$, provided k is chosen to satisfy

$$k^2 = \gamma 2^{l\gamma} l_j^{m^- - 1 + \frac{n(m^- - m)}{2}} M_{l+1}^{\frac{(n+2)(m^+ - m^-)}{2}} (\theta,\tau)\rho^{-n-2} \iint\limits_{Q_{l+1}} F^2(u) dxdt.$$

From this, we obtain

$$M_l^{1-m^- + \frac{n(m-m^-)}{2}} F^2(M_l)$$

$$\leq \gamma(1-\sigma)^{-\gamma} 2^{l\gamma} M_{l+1}^{\frac{(n+2)(m^+ - m^-)}{2}} \rho^{-n-2} \iint\limits_{Q_{l+1}} F^2(u) dxdt.$$

Denoting $M_l^{\frac{1-m^-}{2} + \frac{n(m-m^-)}{4}} F(M_l) = M_l^{\frac{a}{2}} F(M_l) = \Psi_l$, we have

$$\Psi_l^2 \leq \gamma(1-\sigma)^{-\gamma} 2^{l\gamma} \Psi_{l+1} M_{l+1}^{\frac{(n+2)(m^+ - m^-)}{2} - \frac{a}{2}} \rho^{-n-2} \iint\limits_{Q_{l+1}} F(u) dxdt$$

$$\le \varepsilon\Psi_{l+1}^2 + \frac{1}{\varepsilon}(1-\sigma)^{-\gamma}\gamma 2^{l\gamma}(M(\theta,\tau))^{(n+2)(m^+-m^-)-a}$$

$$\times\rho^{-2(n+2)}\left(\iint\limits_{Q_{l+1}} F(u)dxdt\right)^2.$$

By iteration, we get

$$\Psi^2(u(\bar{x},\bar{t})) \le \Psi_0^2 \le \varepsilon^l\Psi_l^2 + \frac{1}{\varepsilon}\gamma(1-\sigma)^{-\gamma}\sum_{i=0}^{l-1}(\varepsilon 2^\gamma)^i$$

$$\times(M(\theta,\tau))^{(n+2)(m^+-m^-)-a}\rho^{-2(n+2)}\left(\iint\limits_{Q_{l+1}} F(u)dxdt\right)^2.$$

We choose $\varepsilon = 2^{-\gamma-1}$. Passing to the limit as $l\to\infty$, we obtain

$$(u(\bar{x},\bar{t}))^{1-m^-+\frac{n(m-m^-)}{2}}F(u(\bar{x},\bar{t}))$$

$$\le \gamma(1-\sigma)^{-\gamma}\rho^{-n-2}(M(\theta,\tau))^{\frac{(n+2)(m^+-m^-)}{2}}\iint\limits_{Q_{\frac{\eta}{2},\frac{s}{2}}(\bar{x},\bar{t})} f(u)u^{m^-}dxdt. \tag{2.5}$$

To estimate the integral on the right-hand side of (2.5), we test the integral identity by $\varphi = u^{m^-}\zeta^2$. Using conditions (1.4) and the Hölder inequality, we obtain

$$\iint\limits_{Q_{\frac{\eta}{2},\frac{s}{2}}(\bar{x},\bar{t})} f(u)u^{m^-}\zeta^2dxdt \le \gamma\iint\limits_{Q_{\frac{\eta}{2},\frac{s}{2}}(\bar{x},\bar{t})} u^{m^-+1}|\zeta_t|\zeta dxdt + \gamma\sum_{i=1}^n\iint\limits_{Q_{\frac{\eta}{2},\frac{s}{2}}(\bar{x},\bar{t})} u^{m_i+m^-}|\zeta_{x_i}|^2dxdt$$

$$\le \gamma\rho^{-2}M^{m^++m^-}(\theta,\tau)|Q_{\frac{\eta}{2},\frac{s}{2}}(\bar{x},\bar{t})| \le \gamma\rho^nM^{m^-+1+\frac{m-m^+}{2}n}(\theta,\tau). \tag{2.6}$$

Let $(\bar{x},\bar{t})$ be an arbitrary point in $Q_{\sigma\theta,\sigma\tau}(x^{(0)},t^{(0)})$. From (2.5) and (2.6), we arrive at

$$(M(\sigma\theta,\sigma\tau))^{1-m^-+\frac{n(m-m^-)}{2}}F(M(\sigma\theta,\sigma\tau))$$

$$\le \gamma(1-\sigma)^{-\gamma}\rho^{-2}(M(\theta,\tau))^{m^++1+\frac{n(m-m^-)}{2}}. \tag{2.7}$$

For $j = 0,1,2\ldots$, we define the sequences $\{\sigma_j\},\{\theta_j\},\{\tau_j\},\{M_j\}$ by $\sigma_j := \frac{1-2^{-j-1}}{1-2^{-j-2}}$, $\theta_j := (\theta_{1j},\theta_{2j},\ldots,\theta_{nj})$, $\theta_{ij} = \theta_i\left(1+\frac{1}{2}+\cdots+\frac{1}{2^j}\right)$, $i = \overline{1,n}$, $\tau_j = \tau\left(1+\frac{1}{2}+\cdots+\frac{1}{2^j}\right)$, $M_j := \sup\limits_{Q_{\theta_j,\tau_j}(x^{(o)})}$, $\Gamma(M_j) = \left[\frac{F(M_j)}{M_j^{m^++m^-}}\right]^{\frac{1}{m^++1+\frac{n(m-m^-)}{2}}}$.

We write (2.7) for the pair of boxes $Q_{\theta_j,\tau_j}(x^{(0)},t^{(0)})$ and $Q_{\theta_{j+1},\tau_{j+1}}(x^{(0)},t^{(0)})$. This gives

$$M_j\Gamma(M_j) \le \gamma(1-\sigma)^{-\gamma}2^{j\gamma}\rho^{\frac{-2}{m^++1+\frac{n(m-m^-)}{2}}}M_{j+1}.$$

We use the following inequality, which is an immediate consequence of our choice of Γ:

$$\Gamma(u)v \le \varepsilon^{-1}\Gamma(u)u + \Gamma(\varepsilon v)v, \qquad \varepsilon,u,v>0. \tag{2.8}$$

Indeed, if $v \le \varepsilon^{-1}u$, then $\Gamma(u)v \le \varepsilon^{-1}\Gamma(u)u$, and if $v \ge \varepsilon^{-1}u$, then $\Gamma(u)v \le \Gamma(\varepsilon v)v$. In both cases, (2.8) holds.

If $\varepsilon \in (0,1) and \mu = \frac{\beta}{m^+ + 1 + \frac{n(m-m^-)}{2}}$, then

$$\Gamma(M_l) \le \Gamma(\varepsilon M_{l+1}) + \frac{1}{\varepsilon}\frac{\Gamma(M_l)M_l}{M_{l+1}}$$

$$\le \Gamma(\varepsilon M_{l+1}) + \varepsilon^{-1}\gamma(1-\sigma)^{-\gamma}2^{l\gamma}\rho^{\overline{\frac{-2}{m^+ + 1 + \frac{n(m-m^-)}{2}}}}$$

$$\le \varepsilon^{\mu}\Gamma(M_{l+1}) + \varepsilon^{-1}\gamma(1-\sigma)^{-\gamma}2^{l\gamma}\rho^{\overline{\frac{-2}{m^+ + 1 + \frac{n(m-m^-)}{2}}}}.$$

By iteration, we get

$$\Gamma(M_0) \le \varepsilon^{l\mu}\Gamma(M_{i+1}) + \varepsilon^{-1}\gamma(1-\sigma)^{-\gamma}\sum_{i=0}^{l}(\varepsilon^{i\mu}2^{i\gamma})\rho^{\overline{\frac{-2}{m^+ + 1 + \frac{n(m-m^-)}{2}}}}.$$

We choose $\varepsilon^{\mu} = 2^{-\gamma-1}$ and pass to the limit as $l \to \infty$. We obtain

$$\Gamma(u(x^{(0)}, t^{(0)})) \le \gamma(1-\sigma)^{-\gamma}\rho^{\overline{\frac{-2}{m^+ + 1 + \frac{n(m-m^-)}{2}}}}.$$

Return to the previous notation

$$F(u(x^{(0)}, t^{(0)})) \le \gamma(1-\sigma)^{-\gamma}(M(\theta,\tau))^{m^+ + m^-}\rho^{-2}. \tag{2.9}$$

Thus, Theorem 1.2 is proved. □

3. Proof of Theorem 1.1

3.1. Pointwise estimates of solutions

Let

$$Q_r = \left\{ (x,t) \in \Omega_T : \left(t^{\frac{\kappa(\lambda)}{\kappa_1(\lambda)}} + \sum_{i=1}^{n} |x_i|^{\frac{\kappa_i(\lambda)}{\kappa_1(\lambda)}} \right)^{\kappa_1(\lambda)} < r, \right\},$$

where $\kappa(\lambda) = \frac{1}{2+(n-\lambda)(m-1)}$, $\kappa_i(\lambda) = \frac{2}{2+(n-\lambda)(m-m_i)}$, $i = \overline{1,n}$, $\lambda = n - \frac{2}{q-m}$. For $0 < r < \rho < \frac{R_0}{2}$ $(R_0 : Q_{R_0} \subset \Omega_T)$ we set $M(r) = \sup\limits_{Q_{R_0} \setminus Q_r} u(x,t)$ and $u_{2\rho} = u(x,t) - M(2\rho) \le M\left(\frac{\rho}{2}\right) - M(2\rho)$ for $(x,t) \in Q_{R_0} \setminus Q_{\frac{\rho}{2}}$. For fixed $k > 0$ and $j = 0, 1, \ldots$, we set $\rho_j = \frac{\rho}{4}\left(1 + \frac{1}{2^j}\right)$, $k_j = k(1-2^{-j})$, $A_{k_j,j} = \{(x,t) \in Q_{\rho_j} : u_{2\rho} > k_j\}$. Let $\zeta_j \in C^{\infty}\left(Q_{\frac{\rho_{j+1}+\rho_j}{2}}\right)$, $0 \le \zeta_j \le 1$, $\zeta_j = 1$ outside Q_{ρ_j}, $\zeta_j = 0$ in $Q_{\rho_{j+1}}$, and $\left|\frac{\partial \zeta_j}{\partial t}\right| \le \gamma 2^{j\gamma}\rho^{-\frac{1}{\kappa(\lambda)}}$, $\left|\frac{\partial \zeta_j}{\partial x_i}\right| \le \gamma 2^{j\gamma}\rho^{-\frac{2}{\kappa_i(\lambda)}}$, $i = \overline{1,n}$. Let i_0 be the number such that $m_i \le 1,\ i = 1, \ldots i_0$ and $m_i > 1,\ i = i_0+1, \ldots n$, $m' = \frac{1}{n}\sum\limits_{i=1}^{i_0} m_i$, $m'' = \frac{1}{n}\sum\limits_{i=i_0+1}^{n} m_i$. Note that $i_0 = 0$, if $m_i > 1,\ i = \overline{1,n}$, and $i_0 = n$, if $m_i \le 1,\ i = \overline{1,n}$.

Testing identity (1.4) by $\varphi = (u_{2\rho} - k_j)_+ \zeta_j^2$ and using conditions (1.4), we obtain

$$\operatorname{ess\,sup} \int\limits_{A_{k_j,j}(t)} (u_{2\rho} - k_j)_+^2 \zeta_j dx + \sum_{i=1}^{i_0} M^{m_i - 1} \left(\frac{\rho}{2}\right) \iint\limits_{A_{k_j,j}} |u_{x_i}|^2 \zeta_j^2 dxdt$$

$$+ \sum_{i=i_0+1}^{n} k_j^{m_i - 1} \iint\limits_{A_{k_j,j}} |u_{x_i}|^2 \zeta_j^2 dxdt + \iint\limits_{A_{k_j,j}} (u_{2\rho} - k_j)_+ u^q \zeta_j^2 dxdt$$

$$\leq \gamma \left(M^2 \left(\frac{\rho}{2}\right) \rho^{-\frac{1}{\kappa(\lambda)}} + \sum_{i=1}^{n} M^{m_i+1} \left(\frac{\rho}{2}\right) \rho^{-\frac{2}{\kappa_i(\lambda)}} \right) |A_{k_j,j}|. \tag{3.1}$$

By Lemma 2.2, the Hölder inequality, and estimate (3.1), we obtain

$$Y_{j+1} = \iint\limits_{A_{k_{j+1},j+1}} (u_{2\rho} - k_{j+1})_+^2 dxdt$$

$$\leq \left|A_{k_{j+1},j+1}\right|^{\frac{2}{n+2}} \left(\iint\limits_{A_{k_{j+1},j+1}} ((u_{2\rho} - k_{j+1})_+ \zeta_j)^{2+\frac{4}{n}} dxdt \right)^{\frac{n}{n+2}}$$

$$\leq \left|A_{k_{j+1},j+1}\right|^{\frac{2}{n+2}} \operatorname*{ess\,sup}_{0<t<T} \left(\int\limits_{A_{k_{j+1},j+1}(t)} (u_{2\rho} - k_{j+1})_+^2 \zeta_j^2 dx \right)^{\frac{2}{n+2}}$$

$$\times \left(\int\limits_0^t \prod_{i=1}^{n} \left(\int\limits_{A_{k_{j+1},j+1}(t)} |((u_{2\rho} - k_{j+1})_+ \zeta_j)_{x_i}|^2 \, dx \right)^{\frac{1}{n}} d\tau \right)^{\frac{n}{n+2}}$$

$$\leq \gamma M^{\frac{(1-m')i_0}{n+2}} \left(\tfrac{\rho}{2}\right) k_{j+1}^{\frac{(1-m'')(n-i_0)}{n+2}}$$

$$\times \left(M^2 \left(\tfrac{\rho}{2}\right) \rho^{-\frac{1}{\kappa(\lambda)}} + \sum_{i=1}^{n} M^{m_i+1} \left(\tfrac{\rho}{2}\right) \rho^{-\frac{2}{\kappa_i(\lambda)}} \right) |A_{k_{j+1},j+1}|^{1+\frac{2}{n+2}}.$$

In view of the evident inequality $(u_{2\rho} - k_j)_+ \geq \frac{k}{2^{j+1}}$ on $A_{k_{j+1},j}$, we obtain the estimate

$$Y_{j+1} \leq \gamma 2^{j\gamma} M^{\frac{(1-m')i_0}{n+2}} \left(\tfrac{\rho}{2}\right) k_{j+1}^{\frac{(1-m'')(n-i_0)}{n+2}} \left(M^2 \left(\tfrac{\rho}{2}\right) \rho^{-\frac{1}{\kappa(\lambda)}} \right.$$

$$\left. + \sum_{i=1}^{n} M^{m_i+1} \left(\frac{\rho}{2}\right) \rho^{-\frac{2}{\kappa_i(\lambda)}} \right) Y_j^{1+\frac{2}{n+2}}. \tag{3.2}$$

It follows from Lemma 2.3 that

$$(M(\rho) - M(2\rho))^{\frac{(m''-1)(n-i_0)}{2}+n+4} \leq \gamma 2^{j\gamma} M^{\frac{(1-m')i_0}{n+2}} \left(\frac{\rho}{2}\right)$$

$$\times\left(M^2\left(\frac{\rho}{2}\right)\rho^{-\frac{1}{\kappa(\lambda)}}+\sum_{i=1}^{n}M^{m_i+1}\left(\frac{\rho}{2}\right)\rho^{-\frac{2}{\kappa_i(\lambda)}}\right)\iint\limits_{Q_{\frac{\rho}{2}}}u_{2\rho}^2dxdt. \tag{3.3}$$

By the Hölder inequality and Lemma 2.1, we get

$$(M(\rho)-M(2\rho))^{\frac{(m''-1)(n-i_0)}{2}+n+4}\le\gamma 2^{j\gamma}M^{\frac{(1-m')i_0}{n+2}}\left(\frac{\rho}{2}\right)$$

$$\times\left(M^2\left(\frac{\rho}{2}\right)\rho^{-\frac{1}{\kappa(\lambda)}}+\sum_{i=1}^{n}M^{m_i+1}\left(\frac{\rho}{2}\right)\rho^{-\frac{2}{\kappa_i(\lambda)}}\right)$$

$$\times\left\{F_3(r,\lambda)+(F_1(r,\lambda)+F_2(r,\lambda))^{\frac{1}{2}}F_4^{\frac{1}{2}}(r,\lambda)\right\}|Q_{\frac{\rho}{2}}|^{\frac{q-1}{q+1}}. \tag{3.4}$$

Similarly to [11], we obtain the following estimate:

$$M(\rho)-M(2\rho)\le 0.$$

Iterating the last inequality for any $\rho\le\frac{R_0}{2}$, we get

$$M(\rho)\le M(R_0).$$

This proves the boundedness of solutions.

3.2. End of the proof of Theorem 1.1

Let K be a compact subset in Ω, and $\xi=0$ in $\partial\Omega\times(0,T)$ such that $\xi=1$ for $(x,t)\in K\times(0,T)$. Testing (1.5) by $\varphi=u^{m^-}\xi^2\psi_r$, $\psi=\psi_r$, using conditions (1.3), the Young inequality, and the boundedness of u, and passing to the limit $r\to 0$, we get

$$\sup_{0<t<T}\int\limits_K u^{m^-+1}dx+\sum_{i=1}^{n}\int\limits_0^T\int\limits_K u^{m_i+m^--2}|u_{x_i}|^2dxdt+\int\limits_0^T\int\limits_K u^{q+m^-}dxdt\le\gamma. \tag{3.5}$$

Testing (1.5) by $\varphi\psi_r$, using (1.3), and the boundedness of solution, and passing to the limit $r\to 0$, we obtain the integral identity (1.5) with an arbitrary $\varphi\in W_{loc}^{1,2}(0,T;L_{loc}^2(\Omega))\cap L_{loc}^2(0,T;\overset{o}{W}{}_{loc}^{1,2}(\Omega))$ and $\psi\equiv 1$. Thus, Theorem 1.1 is proved. □

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REFERENCES

1. O. V. Besov, V. P. Il'in, and S. M. Nikolskii, *Integral Representations of Functions and Embedding Theorems*, Wiley, New York, 1978.
2. H. Brezis and L. Veron, "Removable singularities for some nonlinear elliptic equations," *Arch. Rational Mech. Anal.*, **75(1)**, 1–6 (1980).
3. H. Brezis and A. Friedman, "Nonlinear parabolic equations involving measure as initial conditions," *J. Math. Pures Appl.*, **62**, 73–97 (1983).

4. N. Fusco and C. Sbordone, “Some remarks on the regularity of minima of anisotropic integrals,” *Comm. PDE*, **18**, 153–167 (1993).

5. S. Kamin and L. A. Peletier, “Source type solutions of degenerate diffusion equations with absorption,” *Isr. J. Math.*, **50**, 219–230 (1985).

6. I. M. Kolodij, “On boundedness of generalized solutions of parabolic differential equations,” *Vestnik Moskov. Gos. Univ.*, **5**, 25–31 (1971).

7. G. Lieberman, “Gradient estimates for anisotropic elliptic equations,” *Adv. Diff. Equat.*, **10**, No. 7, 767–812 (2005).

8. O. A. Ladyzhenskaya and N. N. Ural’tseva, *Linear and Quasilinear Elliptic Equations*, Academic Press, New York, 1968.

9. Yu. V. Namlyeyeva, A. E. Shishkov, and I. I. Skrypnik, “Removable isolated singularities for solutions of doubly nonlinear anisotropic parabolic equations,” *Appl. Analysis*, **89(10)**, No. 4, 1559–1574 (2010).

10. Yu. V. Namlyeyeva, A. E. Shishkov, and I. I. Skrypnik, “Isolated singularities of solutions of quasilinear anisotropic elliptic equations.” *Adv. Nonlin. Studies*, **6**, No. 4, 617–641 (2006).

11. M. A. Shan, “Removability of an isolated singularity for solutions of anisotropic porous medium equation with absorption term,” *J. Math. Sci.*, **222**, No. 6, 741–749 (2017).

12. M. A. Shan and I. I. Skrypnik, “Keller-Osserman a priori estimates and the Harnack inequality for quasilinear elliptic and parabolic equations with absorption term,” *Nonlinear Analysis*, **155**, 97–114 (2017).

13. I. I. Skrypnik, “Local behaviour of solutions of quasilinear elliptic equations with absorption,” *Trudy Inst. Mat. Mekh. Nats. Akad. Nauk Ukrainy*, **9**, 183–190 (2004).

14. I. I. Skrypnik, “Removability of isolated singularities of solutions of quasilinear parabolic equations with absorption,” *Sb. Math.*, **196**, No. 11, 1693–1713 (2005).

15. I. I. Skrypnik, “Removability of an isolated singularity for anisotropic elliptic equations with absorption,” *Mat. Sborn.*, **199**, No. 7, 8–102 (2008).

16. I. I. Skrypnik, “Removability of isolated singularity for anisotropic parabolic equations with absorption,” *Manuscr. Math.*, **140**, 145–178 (2013).

17. I. I. Skrypnik, “Removability of isolated singularities for anisotropic elliptic equations with gradient absorption,” *Isr. J. Math.*, **215**, 163–179 (2016).

18. I. I. Skrypnik, “Removable singularities for anisotropic elliptic equations,” *Isr. J. Math.*, **41**, No. 4, 1127–1145 (2014).

19. L. Véron, *Singularities of Solutions of Second Order Quasilinear Equations*, Longman, Harlow, 1996.

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