Removable isolated singularities for solutions of anisotropic porous medium equation

Article *in* Annali di Matematica Pura ed Applicata · March 2017

DOI: 10.1007/s10231-017-0647-2

Maria Shan
Vasyl` Stus Donetsk National University

Annali di Matematica
DOI 10.1007/s10231-017-0647-2

Removable isolated singularities for solutions of anisotropic porous medium equation

M. A. Shan[1]

Received: 4 July 2016 / Accepted: 23 February 2017

Abstract We study a class of quasilinear parabolic equations with model representative

$$\frac{\partial u}{\partial t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(|u|^{m_i-1}\frac{\partial u}{\partial x_i}\right) = 0, m_i > 1,\ i = 1, \ldots, s,\ \ m_i < 1,\ i = s+1, \ldots, n.$$

We establish the pointwise condition for removability of singularity for solutions of such equations.

Keywords Anisotropic porous medium equation · Removable isolated singularity · Pointwise estimates

Mathematics Subject Classification 35B40

1 Introduction and main results

In this paper, we study solutions to quasilinear parabolic equation in the divergent form

$$u_t - \operatorname{div} A(x, t, u, \nabla u) = b(x, t, u, \nabla u), \quad (x, t) \in \Omega_T \setminus \{(x_0, 0)\}, \tag{1}$$

satisfying a initial condition

$$u(x, 0) = 0 \qquad x \in \Omega \setminus \{x_0\}, \tag{2}$$

where Ω is a bounded domain in R^n, $n \geq 3$, $x_0 \in \Omega$, $0 < T < \infty$. Set $\Omega_T := \Omega \times (0, T)$.

We suppose that

$$\min_{1\leq i\leq n} m_i > 1 - \frac{2}{n}, \quad \max_{1\leq i\leq n} m_i < m + \frac{2}{n}, \tag{3}$$

✉ M. A. Shan
shan_maria@ukr.net

1 Vasyl' Stus Donetsk National University, ul. 600-richya 21, Vinnytsia 21021, Ukraine

Published online: 04 March 2017

where $m = \frac{1}{n}\sum_{i=1}^{n} m_i$.

The functions $A : \Omega_T \times R \times R^n \longrightarrow R^n$ and $b : \Omega_T \times R \times R^n \longrightarrow R^n$ are such that $A(\cdot,\cdot,u,\varsigma)$, $b(\cdot,\cdot,u,\varsigma)$ are Lebesque measurable for all $u \in R, \varsigma \in R^n$, and $A(x,t,\cdot,\cdot)$, $b(x,t,\cdot,\cdot)$ are continuous for almost all $(x,t) \in \Omega_T$, $A = (a_1, a_2, ...a_n)$. We also assume that the following structure conditions are satisfied:

$$a_i(x,t,u,\varsigma)\varsigma \geq \nu_1 \sum_{i=1}^{n} |u|^{m_i-1}|\varsigma_i|^2$$

$$|a_i(x,t,u,\varsigma)| \leq \nu_2 |u|^{\frac{m_i-1}{2}} \left(\sum_{j=1}^{n} |u|^{m_j-1}|\varsigma_j|^2\right)^{\frac{1}{2}}, \quad i = \overline{1,n},$$

$$|b(x,t,u,\varsigma)| \leq \nu_2 |u|^{\frac{m-1}{2}} \left(\sum_{j=1}^{n} |u|^{m_j-1}|\varsigma_j|^2\right)^{\frac{1}{2}} \tag{4}$$

with some positive constant ν_1, ν_2.

The qualitative behavior of solutions to quasilinear elliptic and parabolic equations near the point singularity was investigated by many authors starting from the seminal papers of Serrin [1,2] and Aronson and Serrin [3]. Subsequently various extensions of these results have been obtained by many authors. We refer to the monographs by [4,5] for an account of these results.

During the last decade there has been growing interest and substantial development in theory of second-order quasilinear elliptic and parabolic equations with nonstandard growth conditions.

Some results of [6–11] we mention here.

The basic prototype of such equation is

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i}\right) = 0, \tag{5}$$

$$\frac{\partial u}{\partial t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i}\right) = 0, \tag{6}$$

$$\frac{\partial u}{\partial t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(|u|^{m_i-1} \frac{\partial u}{\partial x_i}\right) = 0. \tag{7}$$

The examples constructed by Giaquinta [12] and Marcellini [13] show that (5)–(7) may have unbounded solutions if p_i are too far apart. Local boundedness of solutions to Eqs. (5)–(7) has been obtained in [14–16]. The question of removability of isolated singularity for solutions to anisotropic elliptic Eq. (5) or for Eqs. (6), (7) with $p_i > 2$, $m_i > 1$ was studied in [17,18].

We are interested here in the removability result for the Eq. (7). Note that the important difference between this work and [17] is the following condition

$$m_i > 1,\ i = \overline{1,s},\ \ m_i < 1,\ i = \overline{s+1,n}.$$

Let

$$D(r) = \left\{ (x,t) \in \Omega_T : \sum_{i=1}^{n} \left(\frac{|x_i - x_i^{(0)}|}{r^{k_i}} \right)^2 + \frac{t}{r^k} \leq 1 \right\}, \tag{8}$$

where

$$k = n(m-1) + 2, \quad k_i = \frac{2 + n(m - m_i)}{2}. \tag{9}$$

We formulate the removability result in the form of behavior of the function

$$M(r) = \operatorname{ess\,sup}\{| \, u(x,t) \, | : (x,t) \in D(R_0) \setminus D(r)\}, \tag{10}$$

where R_0 is some sufficiently small fixed positive number such that $D(R_0) \subset \Omega_T$. It follows from [16] that $M(r)$ is finite number for any $r > 0$.

We will write $V_m(\Omega_T)$ for the class of functions $\varphi \in C(0, T, L_2(\Omega))$ with $\sum_{i=1}^{n} \iint_{\Omega_T} |\varphi|^{m_i - 1} \left| \frac{\partial \varphi}{\partial x_i} \right|^2 \mathrm{d}x\mathrm{d}t < \infty$ and $\overset{o}{V}_m(\Omega_T) = \{\varphi \in V_m(\Omega_T) : \varphi|_{(0,T)\times\partial\Omega} = 0\}$. Before formulating the main results, let us remind the reader the definition of a weak solution to (1), (2). We say that u is a weak solution to the problems (1), (2) if for an arbitrary function $\psi \in C^\infty(\Omega_T)$, vanishing in a neighborhood of $\{(x_0, 0)\}$, we have an inclusion $u\psi \in V_m(\Omega_T)$ and the integral identity

$$\int_\Omega (u\varphi\psi)(\cdot,\tau)\mathrm{d}x - \int_0^\tau \int_\Omega u \frac{\partial(\varphi\psi)}{\partial t} \mathrm{d}x\mathrm{d}t + \sum_{i=1}^{n} \int_0^\tau \int_\Omega a_i \left(x, t, u, \frac{\partial u}{\partial x} \right) \frac{\partial(\varphi\psi)}{\partial x_i} \mathrm{d}x\mathrm{d}t - \int_0^\tau \int_\Omega b\left(x, t, u, \frac{\partial u}{\partial x} \right) \varphi\psi \, \mathrm{d}x\mathrm{d}t = 0 \tag{11}$$

holds for every $\varphi \in \overset{o}{V}_m(\Omega_T)$ and for all $\tau \in (0, T)$.

Theorem 1 *Assume that conditions* (3), (4) *are fulfilled. Let u be a weak solution of the problem* (1), (2). *Then the singularity of solution u at the point $\{(x_0, 0)\}$ is removable if*

$$\lim_{r \to 0} M(r) r^n = 0. \tag{12}$$

The proof of the Theorem 1 is based on the following two lemmas.

Lemma 1 *Let the conditions of Theorem 1 be fulfilled. Then there exists a positive constants K_1, β depending only on $\nu_1, \nu_2, n, m_1, \ldots, m_n, m, R_0$ such that the following inequality is true*

$$M(\rho) \leq K_1 \rho^{-n+\beta}, \quad 0 < \rho < R_0. \tag{13}$$

Lemma 2 *Let the conditions of Theorem 1 be fulfilled. Then there exists a positive constant K_2 depending only on $\nu_1, \nu_2, n, m_1, \ldots, m_n, m, R_0$ such that the following inequality is valid*

$$|u(x,t)| \leq K_2, \quad \forall (x,t) \in D\left(\frac{R_0}{2}\right). \tag{14}$$

The rest of the paper contains the proof of the above theorem and lemmas.

2 The integral estimates of solutions

We suppose that

$$\lim_{r\to 0} M(r) = \infty \tag{15}$$

and fixe sufficiently small the number $R_0 : M(R_0) \geq 1$. For every $\rho : 2\rho \leq R_0$ let us use the following notations: $u_{2\rho}(x,t) = (u(x,t) - M(2\rho))_+$, $E_{2\rho} = \{(x,t) \in D(2\rho) : u(x,t) > M(2\rho)\}$.

We further denote by γ constant, depending only on the known parameters $\nu_1, \nu_2, n, m_1, \ldots, m_n, m, R_0$, which may vary from line to line.

Let $\eta_r \in C^\infty(\Omega_T)$ be the cutoff function such that (i) $0 \leq \eta_r(x,t) \leq 1$ in Ω_T, (ii) $\eta_r \equiv 0$ in $D(r)$, $\eta_r \equiv 1$ outside $D(2r)$, (iii)$\left|\frac{\partial \eta_r}{\partial t}\right| \leq \gamma r^{-k}$, $\left|\frac{\partial \eta_r}{\partial x_i}\right| \leq \gamma r^{-k_i}$, where k, k_i, $i = \overline{1,n}$ are defined by (9).

Lemma 3 *Let the conditions of Theorem 1 be fulfilled. Then the following inequality is valid for every* $r : 0 < r < \rho < R_0$

$$\operatorname*{ess\,sup}_{0<t<T} \int_\Omega u^{\theta+1}(x,t)\eta_r^2(x,t)dx + \sum_{i=1}^n \iint_{E_{2\rho}} u^{m_i-1} u_{2\rho}^{\theta-1} \left|\frac{\partial u}{\partial x_i}\right|^2 \eta_r^2 dxdt$$
$$\leq \gamma M^\theta(r), \tag{16}$$

where $\theta \in \left(0, \min\limits_{1\leq i\leq n} m_i\right)$.

Proof Testing identity (11) by

$$\varphi(x,t) = u_{2\rho}^\theta(x,t)\eta_r(x,t), \quad \psi(x,t) = \eta_r(x,t).$$

Applying conditions (4) and Young's inequality, we obtain:

$$\operatorname*{ess\,sup}_{0<t<T} \int_\Omega u_{2\rho}^{\theta+1}(x,t)\eta_r^2(x,t)\mathrm{d}x + \sum_{i=1}^n \iint_{E_{2\rho}} u^{m_i-1} u_{2\rho}^{\theta-1} \left|\frac{\partial u}{\partial x_i}\right|^2 \eta_r^2 \mathrm{d}x\mathrm{d}t$$
$$\leq \gamma \iint_{E_{2\rho}} u_{2\rho}^{\theta+1} \left|\frac{\partial \eta_r}{\partial t}\right| \eta_r \mathrm{d}x\mathrm{d}t + \gamma \sum_{i=1}^n \iint_{E_{2\rho}} u^{m_i-1} u_{2\rho}^{\theta+1} \left|\frac{\partial \eta_r}{\partial x_i}\right|^2 \eta_r \mathrm{d}x\mathrm{d}t$$
$$+\gamma \iint_{E_{2\rho}} u^{m-1} u_{2\rho}^{\theta+1} \eta_r^2 \mathrm{d}x\mathrm{d}t. \tag{17}$$

Using the definitions of $M(r)$ and $\eta_r(x,t)$, we get:

$$\gamma \iint_{E_{2\rho}} u_{2\rho}^{\theta+1} \left|\frac{\partial \eta_r}{\partial t}\right| \eta_r \mathrm{d}x\mathrm{d}t + \gamma \sum_{i=1}^n \iint_{E_{2\rho}} u^{m_i-1} u_{2\rho}^{\theta+1} \left|\frac{\partial \eta_r}{\partial x_i}\right|^2 \eta_r \mathrm{d}x\mathrm{d}t$$
$$\leq \gamma M^\theta(r) \left(M(r)r^{-k} + \sum_{i=1}^n M^{m_i}(r) r^{-2k_i}\right) |D(2r) \setminus D(r)| \leq \gamma M^\theta(r) \tag{18}$$

It is easy to see that the last term on the right-hand side of (17) can be estimated as follows:

$$\iint\limits_{E_{2\rho}} u^{m-1}u_{2\rho}^{\theta+1}\eta_r^2\mathrm{d}x\mathrm{d}t \le \gamma M^\theta(r)\iint\limits_{E_{2\rho}}\left(\sum_{i=1}^n |x_i|^{\frac{1}{k_i}}+t^{\frac{1}{k}}\right)^{-nm}\mathrm{d}x\mathrm{d}t \le \gamma M^\theta(r) \quad (19)$$

Combining estimates (17)–(19), we derive the required estimate (16). □

Introduce the following notations:

$$\Phi_{\rho,2\rho}(u(x,t)) = \min\{u_{2\rho}, M(\rho)-M(2\rho)\},$$
$$E(\rho,2\rho) = \{(x,t)\in\Omega_T : 0 < u_{2\rho}(x,t) < M(\rho)-M(2\rho)\},$$
$$\varepsilon(r) = M(r)r^n + r^2(M(r)r^n)^m + \sum_{i=1}^n (M(r)r^n)^{m_i}.$$

Lemma 4 *Let the conditions of Theorem 1 be fulfilled. Then the following estimate is true*

$$\operatorname*{ess\,sup}_{0<t<T}\int\limits_\Omega \Phi_{\rho,2\rho}^2(u(x,t))\eta_r^2(x,t)dx + \sum_{i=1}^n\iint\limits_{E(\rho,2\rho)} u^{m_i-1}\left|\frac{\partial u}{\partial x_i}\right|^2\eta_r^2 dxdt$$
$$\le \gamma(M(\rho)-M(2\rho))\sum_{i=1}^n\iint\limits_{E_\rho} u^{\frac{m-1}{2}}\left(u^{m_i-1}\left|\frac{\partial u}{\partial x_i}\right|^2\right)^{\frac12}\eta_r^2 dxdt$$
$$+\gamma\varepsilon(r)(M(\rho)-M(2\rho)) + \gamma(M(\rho)-M(2\rho))^{1-\delta_1}, \quad (20)$$

where $\delta_1\in\left(0,\frac{2}{n}\right)$, $0<r<\rho<R$.

Proof Testing (11) by

$$\varphi(x,t) = \Phi_{\rho,2\rho}(u(x,t))\eta_r(x,t), \qquad \psi(x,t) = \eta_r(x,t).$$

Using structural inequalities (4), we have

$$\operatorname*{ess\,sup}_{0<t<T}\int\limits_\Omega \Phi_{\rho,2\rho}^2(u(x,t))\eta_r^2(x,t)\mathrm{d}x + \iint\limits_{E_{(\rho,2\rho)}}\sum_{i=1}^n u^{m_i-1}\left|\frac{\partial u}{\partial x_i}\right|^2\eta_r^2\mathrm{d}x\mathrm{d}t$$
$$\le \gamma(M(\rho)-M(2\rho))\sum_{i=1}^n\iint\limits_{E_\rho} u^{\frac{m_i-1}{2}}\left(\sum_{j=1}^n u^{m_j-1}\left|\frac{\partial u}{\partial x_j}\right|^2\right)^{\frac12}\left|\frac{\partial\eta_r}{\partial x_i}\right|\eta_r\mathrm{d}x\mathrm{d}t$$
$$+\gamma\sum_{i=1}^n\iint\limits_{E_\rho}\Phi_{\rho,2\rho}(u(x,t))u^{\frac{m-1}{2}}\left(u^{m_i-1}\left|\frac{\partial u}{\partial x_i}\right|^2\right)^{\frac12}\eta_r^2\mathrm{d}x\mathrm{d}t$$
$$+\gamma\iint\limits_{E_\rho}\Phi_{\rho,2\rho}^2(u(x,t))\left|\frac{\partial\eta_r}{\partial t}\right|\eta_r\mathrm{d}x\mathrm{d}t. \quad (21)$$

Applying Hölder's inequality and Lemma 3 to the first term on the right-hand side of estimate (21), we derive

$$\sum_{i=1}^{n}\iint_{E_\rho} u^{\frac{m_i-1}{2}}\left(\sum_{j=1}^{n} u^{m_j-1}\left|\frac{\partial u}{\partial x_j}\right|^2\right)^{\frac{1}{2}}\left|\frac{\partial \eta_r}{\partial x_i}\right|\eta_r^2 \mathrm{d}x\mathrm{d}t$$
$$\leq \gamma\sum_{i=1}^{n} M^{\frac{m_i-\theta}{2}}(r) r^{\frac{n+k}{2}-k_i}\left(\sum_{j=1}^{n}\iint_{E_\rho} u^{m_j-1}u_{2\rho}^{\theta-1}\left|\frac{\partial u}{\partial x_j}\right|^2\eta_r^2\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}}$$
$$\leq \gamma\sum_{i=1}^{n}\left[M(r)r^n\right]^{\frac{m_i}{2}} \leq \gamma\varepsilon(r). \tag{22}$$

The second term on the right-hand side of (21) can be represented in the following way:

$$\sum_{i=1}^{n}\iint_{E_\rho} \Phi_{\rho,2\rho}(u(x,t))u^{\frac{m-1}{2}}\left(u^{m_i-1}\left|\frac{\partial u}{\partial x_i}\right|^2\right)^{\frac{1}{2}}\eta_r^2\mathrm{d}x\mathrm{d}t$$
$$\leq \sum_{i=1}^{n}\iint_{E_\rho} \Phi_{\rho,2\rho}(u(x,t))u^{\frac{m-1}{2}}\left(u^{m_i-1}\left|\frac{\partial u}{\partial x_i}\right|^2\right)^{\frac{1}{2}}\eta_r^2\mathrm{d}x\mathrm{d}t$$
$$+\sum_{i=1}^{n}\iint_{E_{(\rho,2\rho)}} \Phi_{\rho,2\rho}(u(x,t))u^{\frac{m-1}{2}}\left(u^{m_i-1}\left|\frac{\partial u}{\partial x_i}\right|^2\right)^{\frac{1}{2}}\eta_r^2\mathrm{d}x\mathrm{d}t. \tag{23}$$

Estimating the second integral on the right-hand of (23), we have

$$\sum_{i=1}^{n}\iint_{E_{(\rho,2\rho)}} \Phi_{\rho,2\rho}(u(x,t))u^{\frac{m-1}{2}}\left(u^{m_i-1}\left|\frac{\partial u}{\partial x_i}\right|^2\right)^{\frac{1}{2}}\eta_r^2\mathrm{d}x\mathrm{d}t$$
$$\leq \gamma\sum_{i=1}^{n}\iint_{E_{(\rho,2\rho)}} u^{m_i-1}\left|\frac{\partial u}{\partial x_i}\right|^2\eta_r^2\mathrm{d}x\mathrm{d}t+\gamma(M(\rho)-M(2\rho))^{1-\delta_1}. \tag{24}$$

Combining estimates (21)–(24), we obtain (20). This completes the proof of the Lemma 4. □

Lemma 5 *Let the conditions of Theorem 1 be fulfilled. Then the following estimate*

$$\sum_{i=1}^{n}\iint_{E_\rho} u_{2\rho}^{-q}u^{m_i-1}\left|\frac{\partial u}{\partial x_i}\right|^2\eta_r^2 dxdt$$
$$\leq \gamma(M(\rho)-M(2\rho))^{2(1-q)}+\gamma(M(\rho)-M(2\rho))^{1-q}\varepsilon(r) \tag{25}$$

holds with $1<q<1+\min\left[\frac{2}{n},\frac{1}{2}\right]$.

Proof Testing identity (11) by

$$\varphi=\left([M(\rho)-M(2\rho)]^{1-q}-[\max(u_{2\rho},M(\rho)-M(2\rho))]^{1-q}\right)_+\eta_r,\qquad \psi=\eta_r.$$

Using conditions (4) and Young's inequality, we obtain

$$\sum_{i=1}^{n}\iint\limits_{E_\rho}\sum_{i=1}^{n} u_{2\rho}^{-q} u^{m_i-1}\left|\frac{\partial u}{\partial x_i}\right|^2 \eta_r^2 \mathrm{d}x\mathrm{d}t$$

$$\leq \gamma(M(\rho)-M(2\rho))^{1-q}\sum_{i=1}^{n}\iint\limits_{E_\rho} u^{\frac{m_i-1}{2}}\left|\frac{\partial \eta_r}{\partial x_i}\right|\left(\sum_{j=1}^{n} u^{m_j-1}\left|\frac{\partial u}{\partial x_j}\right|^2\right)^{\frac{1}{2}} \eta_r \mathrm{d}x\mathrm{d}t$$

$$+\gamma(M(\rho)-M(2\rho))^{1-q}\sum_{i=1}^{n}\iint\limits_{E_\rho} u^{\frac{m-1}{2}}\left(\sum_{j=1}^{n} u^{m_j-1}\left|\frac{\partial u}{\partial x_j}\right|^2\right)^{\frac{1}{2}} \eta_r^2 \mathrm{d}x\mathrm{d}t$$

$$+\gamma\,(M(\rho)-M(2\rho))^{2(1-q)}\iint\limits_{E_\rho}\eta_r\left|\frac{\partial \eta_r}{\partial t}\right|\mathrm{d}x\mathrm{d}t. \tag{26}$$

Estimating the terms on the right-hand side of (26), we obtain

$$\sum_{i=1}^{n}\iint\limits_{E_\rho} u^{\frac{m_i-1}{2}}\left|\frac{\partial \eta_r}{\partial x_i}\right|\left(\sum_{j=1}^{n} u^{m_j-1}\left|\frac{\partial u}{\partial x_j}\right|^2\right)^{\frac{1}{2}} \eta_r \mathrm{d}x\mathrm{d}t \leq$$

$$\leq \left(\sum_{i=1}^{n}\iint\limits_{E_\rho} u^{m_i-1}u^{1-\theta}\left|\frac{\partial \eta_r}{\partial x_i}\right|^2 \eta_r^2 \mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}}\times$$

$$\times\left(\sum_{j=1}^{n}\iint\limits_{E_\rho} u^{m_j-1}u^{\theta-1}\left|\frac{\partial u}{\partial x_j}\right|^2 \eta_r^2 \mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}} \leq \gamma r^{\frac{n\theta}{2}}\left(\gamma M^\theta(r)\right)^{\frac{1}{2}} \leq \gamma\varepsilon(r) \tag{27}$$

$$(M(\rho)-M(2\rho))^{1-q}\sum_{i=1}^{n}\iint\limits_{E_\rho} u^{\frac{m-1}{2}}\left(\sum_{j=1}^{n} u^{m_j-1}\left|\frac{\partial u}{\partial x_j}\right|^2\right)^{\frac{1}{2}} \eta_r^2 \mathrm{d}x\mathrm{d}t$$

$$\leq (M(\rho)-M(2\rho))^{2(1-q)}\rho^{2+n(1-q)}+\sum_{j=1}^{n}\iint\limits_{E_\rho} u^{m_j-1}u_{2\rho}^{-q}\left|\frac{\partial u}{\partial x_j}\right|^2 \eta_r^2 \mathrm{d}x\mathrm{d}t. \tag{28}$$

Combining estimates (26)–(28), we derive required inequality (25). □

Combining Lemmas 4 and 5, we obtain:

Lemma 6 *Let the conditions of Theorem 1 be fulfilled. Then the following estimate is true*

$$\operatorname*{ess\,sup}_{0<t<T}\int\limits_{\Omega}\Phi_{\rho,2\rho}^2(u(x,t))\eta_r^2(x,t)dx+\sum_{i=1}^{n}\iint\limits_{E(\rho,2\rho)} u^{m_i-1}\left|\frac{\partial u}{\partial x_i}\right|^2 \eta_r^2 dxdt$$

$$\leq \gamma\varepsilon(r)(M(\rho)-M(2\rho))+\gamma\varepsilon(r)(M(\rho)-M(2\rho))^{2-q}$$
$$+\gamma(M(\rho)-M(2\rho))^{1-\delta_1}+\gamma(M(\rho)-M(2\rho))^{1-\delta_2}, \tag{29}$$

where $0<\delta_1,\delta_2<1$.

Proof By the Young's inequality, we have

$$\sum_{i=1}^{n}\iint\limits_{E_\rho} u^{\frac{m-1}{2}}\left(u^{m_i-1}\left|\frac{\partial\eta_r}{\partial x_i}\right|^2\right)^{\frac{1}{2}}\eta_r^2\mathrm{d}x\mathrm{d}t$$
$$\le \gamma\sum_{i=1}^{n}\iint\limits_{E_\rho} u_{2\rho}^{-q}u^{m_i-1}\left|\frac{\partial\eta_r}{\partial x_i}\right|^2\eta_r^2\mathrm{d}x\mathrm{d}t+\gamma\iint\limits_{E_\rho} u_{2\rho}^{q}u^{m-1}\eta_r^2\mathrm{d}x\mathrm{d}t. \tag{30}$$

We estimate the last integral on the right-hand side of (30) similarly to (19)

$$\iint\limits_{E_\rho} u_{2\rho}^{q}u^{m-1}\eta_r^k\mathrm{d}x\mathrm{d}t\le\gamma\rho^{2+n(1-q)}. \tag{31}$$

Combining inequalities (20), (25), (30), (31), we obtain estimate (29). □

Taking into account condition (12), we can pass to the limit as $r\to 0$ in (29) and obtain the following statement.

Remark 1 Let the conditions of Theorem 1 be fulfilled. Then the following estimate is valid

$$\operatorname*{ess\,sup}_{0<t<T}\int\limits_{\Omega}\Phi_{\rho,2\rho}^2(u(x,t))\mathrm{d}x+\sum_{i=1}^{n}\iint\limits_{E_{\rho,2\rho}}\sum_{i=1}^{n}u^{m_i-1}\left|\frac{\partial u}{\partial x_i}\right|^2\mathrm{d}x\mathrm{d}t$$
$$\le\gamma(M(\rho)-M(2\rho))^{1-\delta_1}+\gamma(M(\rho)-M(2\rho))^{1-\delta_2}, \tag{32}$$

where $\delta_1,\delta_2\in(0,1)$.

3 The pointwise estimate of solutions

Let $(\widetilde{x},\widetilde{t})$ be an arbitrary point in $D(R_0)\setminus D(\rho)$. For any $\rho: 0<r<\rho<R$ and any positive h we define the sequences of numbers $\rho_j:=\frac{\rho}{2}(1+2^{-j}),\quad h_j:=h(1-2^{-j}),\ j=\overline{1,n}$ and the families of sets:

$$Q(\rho_j):=\left\{(x,t):\sum_{i=1}^{n}\left(\frac{|x_i-\widetilde{x}_i|}{\rho_j^{k_i}}\right)^2+\frac{|t-\widetilde{t}|}{\rho_j^k}<1\right\},$$
$$A_j:=\left\{(x,t)\in Q(\rho_j):u_{2\rho}>h_j\right\}.$$

We denote by $\zeta_j\in C_0^\infty\left(Q\left(\frac{\rho_j+\rho_{j+1}}{2}\right)\right)$ such that: (i) $\zeta_j(x,t)\equiv 1$ outside $Q(\rho_j)$, $\zeta_j(x,t)\equiv 0$ for $(x,t)\in Q(\rho_{j+1})$; (ii) $\left|\frac{\partial\zeta_j}{\partial t}\right|\le\gamma 2^{jk}\rho^{-k}$, $\left|\frac{\partial\zeta_j}{\partial x_i}\right|\le\gamma 2^{jk_i}\rho^{-k_i}$, $i=\overline{1,n}$.

Testing identity (11) by

$$\varphi=(u_{2\rho}-h_j)_+\zeta_j,\qquad\psi=\zeta_j.$$

From conditions (4) , Young's inequality and properties of the cutoff functions ζ_j, we obtain

$$\begin{aligned}
&\operatorname*{ess\,sup}_{0<t<T} \int\limits_{A_j(t)} (u_{2\rho}-h_j)_+^2\zeta_j^2 \mathrm{d}x + \sum_{i=1}^n \iint\limits_{A_j} u^{m_i-1}\left|\frac{\partial u}{\partial x_i}\right|^2 \zeta_j^2 \mathrm{d}x\mathrm{d}t \\
&\le \gamma \sum_{i=1}^n \iint\limits_{A_j} u^{m_i-1}\left|\frac{\partial \zeta_j}{\partial x_i}\right|^2 (u_{2\rho}-h_j)_+^2 \zeta_j \mathrm{d}x\mathrm{d}t \\
&\quad + \gamma \iint\limits_{A_j} (u_{2\rho}-h_j)^2 \left|\frac{\partial \zeta_j}{\partial t}\right| \zeta_j \mathrm{d}x\mathrm{d}t + \gamma \iint\limits_{A_j} u^{m-1}(u_{2\rho}-h_j)_+^2\zeta_j^2 \mathrm{d}x\mathrm{d}t. \qquad (33)
\end{aligned}$$

Using the definition of $M(\rho)$, we have

$$\begin{aligned}
&\sum_{i=1}^n \iint\limits_{A_j} u^{m_i-1}\left|\frac{\partial \zeta_j}{\partial x_i}\right|^2 (u_{2\rho}-h_j)_+^2 \zeta_j \mathrm{d}x\mathrm{d}t \\
&\le \gamma \sum_{i=1}^n 2^{2jk_i}\rho^{-2k_i} M^{m_i+1}\left(\frac{\rho}{2}\right)|A_j| \le \gamma 2^{j\gamma}\rho^{-n(m+1)-2}|A_j|, \qquad (34)
\end{aligned}$$

$$\begin{aligned}
&\iint\limits_{A_j} (u_{2\rho}-h_j)_+^2 \left|\frac{\partial \zeta_j}{\partial t}\right| \zeta_j \mathrm{d}x\mathrm{d}t \le \gamma 2^{jk}\rho^{-k}M^2\left(\frac{\rho}{2}\right)|A_j| \\
&\le \gamma 2^{jk}\rho^{-n(m+1)-2}|A_j|, \qquad (35)
\end{aligned}$$

$$\iint\limits_{A_j} u^{m-1}(u_{2\rho}-h_j)_+^2\zeta_j^2 \mathrm{d}x\mathrm{d}t \le \gamma M^{m+1}\left(\frac{\rho}{2}\right)|A_j| \le \gamma\rho^{-n(m+1)}|A_j|. \qquad (36)$$

Combining inequalities (34)–(36), we derive the following additional integral estimate

$$\begin{aligned}
&\operatorname*{ess\,sup}_{0<t<T} \int\limits_{A_j(t)} (u_{2\rho}-h_j)_+^2\zeta_j^2 \mathrm{d}x + \sum_{i=1}^n \iint\limits_{A_j} u^{m_i-1}\left|\frac{\partial u}{\partial x_i}\right|^2 \zeta_j^2 \mathrm{d}x\mathrm{d}t \\
&\le \gamma 2^{j\gamma}\rho^{-n(m+1)-2}|A_j|. \qquad (37)
\end{aligned}$$

Let i_0 be the number such that $m_i < 1,\ i = \overline{0, i_0}$ and $m_i > 1,\ i = \overline{i_0+1, n}$. Taking into account that $u \le M(\frac{\rho}{2}),\ u > M(2\rho)+h_j$ on A_j, we obtain the following lemma from (37).

Lemma 7 *Let the conditions of Theorem 1 be fulfilled. Then the following estimate is true*

$$\begin{aligned}
&\operatorname*{ess\,sup}_{0<t<T} \int\limits_{A_j(t)} (u_{2\rho}-h_j)_+^2\zeta_j^2 dx + M^{i_0(m'-1)}\left(\frac{\rho}{2}\right)\sum_{i=1}^{i_0} \iint\limits_{A_j} \left|\frac{\partial u}{\partial x_i}\right|^2 \zeta_j^2 dxdt \\
&+ h_j^{(n-i_0)(m''-1)} \sum_{i=i_0+1}^n \iint\limits_{A_j} \left|\frac{\partial u}{\partial x_i}\right|^2 \zeta_j^2 dxdt \le \gamma 2^{(j+1)\gamma}\rho^{-n(m+1)-2}|A_j|, \qquad (38)
\end{aligned}$$

where $m' = \frac{1}{i_0}\sum\limits_{i=0}^{i_0} m_i,\ \ m'' = \frac{1}{n-i_0}\sum\limits_{i=i_0+1}^{n} m_i.$

Now we are ready to prove Theorem 1. Applying Hölder's inequality, Lemma 9 from the appendix with $\alpha_i = 0,\ i = 1, \ldots, n$, we derive:

$$\iint\limits_{A_{j+1}} (u_{2\rho} - h_{j+1})_+^2 \mathrm{d}x\mathrm{d}t$$

$$\leq \gamma\rho^{-n(m+1)-2} M^{-\frac{i_0(m^{'}-1)}{n+2}} \left(\frac{\rho}{2}\right) h^{-\frac{(n-i_0)(m^{''}-1)}{n+2}-2\left(1+\frac{2}{2+n}\right)} \left(\iint\limits_{A_j} (u_{2\rho} - h_j)^2 \mathrm{d}x\, \mathrm{d}t\right)^{1+\frac{2}{2+n}}.$$

Choosing h from the condition

$$\rho^{-n(m+1)-2+\frac{i_0 n(m^{'}-1)}{n+2}} h^{-\frac{(n-i_0)(m^{''}-1)}{n+2}-2\left(1+\frac{2}{2+n}\right)} \left(\iint\limits_{Q(\rho)} u_{2\rho}^2 \mathrm{d}x\mathrm{d}t\right)^{\frac{2}{2+n}} \leq 1$$

and using Lemma 8, we derive

$$\left(M\left(\frac{\rho}{2}\right) - M(2\rho)\right)^{\frac{1}{2}(n-i_0)(m^{''}-1)+n+4} \leq \gamma\rho^{-(n(m+1)+2)\frac{n+2}{2}+\frac{i_0 n(m^{'}-1)}{2}+k} \sup_{0<t<T} \int\limits_{Q(\rho)} u_{2\rho}^2 \mathrm{d}x. \tag{39}$$

From (39) by Remark 1, we obtain

$$M\left(\frac{\rho}{2}\right) - M(2\rho) \leq \gamma\rho^{-n+\beta}, \tag{40}$$

where

$$\beta = \frac{n\delta}{\frac{1}{2}(n-i_0)(m^{''}-1)+n+4} > 0,\ \ \delta = min\{\delta_1,\ \delta_2\}.$$

Iterating inequality (40), we derive estimate (13). This proves Lemma 1.

4 Proof of Lemma 2

For $j = 0, 1, 2\ldots$ set $\rho_j := \frac{R_0}{2}\left(1+2^{-j}\right)$, $\overline{\rho}_j := \frac{1}{2}\left(\rho_j + \rho_{j+1}\right)$, $h_j = h\left(1-2^{-j}\right)$, $A_{h_j,\rho_j} := \left\{(x,t) \in Q(\rho_j) : u^\lambda > h_j\right\}$, where h is a positive number depending on the known parameters only, which will be specified later, and $0 < \lambda < min\{\min\limits_{1\leq i\leq n} m_i, \frac{\beta}{n-\beta}, \frac{\left(1-\frac{2}{n}\right)\beta+n}{n-\beta}\}$ with β from (13). Let $\xi_j \in C_0^\infty(D(\overline{\rho}_j))$, $0 \leq \xi_j \leq 1$, $\xi_j = 1$ in $D(\rho_{j+1})$ and $\left|\frac{\partial\xi_j}{\partial t}\right| \leq \gamma 2^{j\gamma} R_0^{-k}$, $\left|\frac{\partial\xi_j}{\partial x_i}\right| \leq \gamma 2^{j\gamma} R_0^{-k_i}$, $i = \overline{1,n}$, where k, k_i are defined in (9).

Test (11) by

$$\varphi = \left(u^\lambda - h_{j+1}\right)_+ \xi_j^2 \eta_r, \quad \psi = \eta_r,$$

where η_r was defined in Sect. 2. Using conditions (4) and the Young inequality, we obtain

$$\operatorname*{ess\,sup}_{0<t<T} \int\limits_{A_{h_{j+1},\overline{\rho}_j}(t)} \left(u^{\lambda} - h_{j+1}\right)_+^{\frac{1}{\lambda}+1} \xi_j^2 \eta_r^2 \mathrm{d}x + \sum_{i=1}^{n} \iint\limits_{A_{h_{j+1},\overline{\rho}_j}} u^{m_i+\lambda-2} \left|\frac{\partial u}{\partial x_i}\right|^2 \xi_j^2 \eta_r^2 \mathrm{d}x\mathrm{d}t \le \gamma \sum_{i=1}^{5} J_i, \tag{41}$$

where

$$J_1 = \sum_{i=1}^{n} \iint\limits_{A_{h_{j+1},\overline{\rho}_j}} u^{m_i+\lambda} \left|\frac{\partial \eta_r}{\partial x_i}\right|^2 \xi_j^2 \eta_r \mathrm{d}x\mathrm{d}t,$$

$$J_2 = \iint\limits_{A_{h_{j+1},\overline{\rho}_j}} u^{\lambda+1} \left|\frac{\partial \eta_r}{\partial t}\right| \xi_j^2 \eta_r \mathrm{d}x\mathrm{d}t,$$

$$J_3 = \sum_{i=1}^{n} \iint\limits_{A_{h_{j+1},\overline{\rho}_j}} u^{m_i+\lambda} \left|\frac{\partial \xi_j}{\partial x_i}\right|^2 \xi_j \eta_r^2 \mathrm{d}x\mathrm{d}t,$$

$$J_4 = \iint\limits_{A_{h_{j+1},\overline{\rho}_j}} u^{\lambda+1} \eta_r^2 \xi_j \left|\frac{\partial \xi_j}{\partial t}\right| \mathrm{d}x\mathrm{d}t,$$

$$J_5 = \iint\limits_{A_{h_{j+1},\overline{\rho}_j}} u^{m+\lambda} \xi_j^2 \eta_r^2 \mathrm{d}x\mathrm{d}t.$$

Condition (13) imply that

$$\begin{aligned} J_1 &\le \gamma r^{m_i\beta+\lambda(\beta-n)+n}, \\ J_2 &\le \gamma r^{\beta(\lambda+1)-n\beta}, \\ J_3 + J_4 &\le \gamma 2^{j\gamma} R_0^{-n(m+\lambda)-2} \left|A_{h_{j+1},\overline{\rho}_j}\right|, \\ J_5 &\le \gamma 2^{j\gamma} R_0^{-n(m+\lambda)} \left|A_{h_{j+1},\overline{\rho}_j}\right|. \end{aligned} \tag{42}$$

Combining (41), (42), and passing to the limit as $r \to 0$, we obtain

$$\operatorname*{ess\,sup}_{0<t<T} \int\limits_{A_{h_{j+1},\overline{\rho}_j}(t)} \left(u^{\lambda} - h_{j+1}\right)_+^{\frac{1}{\lambda}+1} \xi_j^2 \mathrm{d}x + \sum_{i=1}^{n} \iint\limits_{A_{h_{j+1},\overline{\rho}_j}} u^{m_i+\lambda-2} \left|\frac{\partial u}{\partial x_i}\right|^2 \xi_j^2 \mathrm{d}x\mathrm{d}t \le \gamma 2^{j\gamma} R_0^{-n(m+\lambda)-2} \left|A_{h_{j+1},\overline{\rho}_j}\right|. \tag{43}$$

From this, setting $\widetilde{q} = \frac{m+\lambda}{\lambda} + \frac{1+\lambda}{\lambda}\frac{2}{n}$, $a = 2(m+\lambda)\max\left(\frac{1}{1+\lambda}, \frac{1}{\min\limits_{1\le i\le n} m_i+\lambda}\right)$, using the Hölder inequality and Lemma 10 from the Appendix with $\alpha_i = \frac{m_i-\lambda}{\lambda} > 0$, $i = \overline{1,n}$, we obtain

$$
\begin{aligned}
Y_{j+1} &= \iint\limits_{A_{h_{j+1},\overline{\rho}_{j+1}}} \left(u^{\lambda}-h_{j+1}\right)_{+}^{\frac{m+\lambda}{\lambda}} \mathrm{d}x\mathrm{d}t \le \iint\limits_{A_{h_{j+1},\overline{\rho}_{j}}} \left(u^{\lambda}-h_{j+1}\right)_{+}^{\frac{m+\lambda}{\lambda}} \xi_j^{a}\mathrm{d}x\mathrm{d}t \\
&\le \left(\iint\limits_{A_{h_{j+1},\overline{\rho}_{j}}} \left(u^{\lambda}-h_{j+1}\right)_{+}^{\widetilde{q}} \xi_j^{\frac{a\lambda\widetilde{q}}{m+\lambda}}\mathrm{d}x\mathrm{d}t \right)^{\frac{m+\lambda}{\lambda\widetilde{q}}} \left|A_{h_{j+1},\overline{\rho}_{j}}\right|^{1-\frac{m+\lambda}{\lambda\widetilde{q}}} \\
&\le \gamma \left(\sup_{0<t<T} \int\limits_{A_{h_{j+1},\overline{\rho}_{j}}(t)} \left(u^{\lambda}-h_{j+1}\right)_{+}^{\frac{1+\lambda}{\lambda}} \xi_j^{2}\mathrm{d}x \right)^{\frac{2}{n}\frac{m+\lambda}{\lambda\widetilde{q}}} \\
&\quad \times \left(\sum_{i=1}^{n} \iint\limits_{A_{h_{j+1},\overline{\rho}_{j}}} \left(\left(u^{\lambda}-h_{j+1}\right)_{+} \xi_j^{\frac{a\lambda}{m+\lambda}} \right)^{\frac{m_i-\lambda}{\lambda}} \left| \frac{\partial}{\partial x_i}\left\{ (u^{\lambda}-h_{j+1})_{+}\xi_j^{\frac{a\lambda}{m+\lambda}} \right\} \right|^{2} \mathrm{d}x\mathrm{d}t \right)^{\frac{m+\lambda}{\lambda\widetilde{q}}} \\
&\quad \times \left|A_{h_{j+1},\overline{\rho}_{j}}\right|^{1-\frac{m+\lambda}{\lambda\widetilde{q}}} \le \gamma 2^{j\gamma} R_0^{-(n(m+\lambda)+2)\frac{m+\lambda}{\lambda\widetilde{q}}\left(1+\frac{2}{n}\right)} \left|A_{h_{j+1},\overline{\rho}_{j}}\right|^{1+\frac{2}{n}\frac{m+\lambda}{\lambda\widetilde{q}}} \\
&\le \gamma 2^{j\gamma} h^{-\frac{m+\lambda}{\lambda}\left(1+\frac{2}{n}\frac{m+\lambda}{\lambda\widetilde{q}}\right)} R_0^{-(n(m+\lambda)+2)\frac{m+\lambda}{\lambda\widetilde{q}}\left(1+\frac{2}{n}\right)} Y_j^{1+\frac{2}{n}\frac{m+\lambda}{\lambda\widetilde{q}}}.
\end{aligned}
\tag{44}
$$

Due to Lemma 8 from the Appendix, this inequality implies that $Y_j \to 0$ as $j \to \infty$ if h satisfies the condition $h^{\frac{m+\lambda}{\lambda}\left(1+\frac{n}{2}\frac{\lambda\widetilde{q}}{m+\lambda}\right)} = \gamma Y_0$, which implies

$$
\begin{aligned}
&\operatorname{ess\,sup}\left\{ |u| : (x,t) \in D\left(\frac{R_0}{2}\right) \right\}^{m+\lambda+\frac{n\lambda\widetilde{q}}{2}} \\
&\quad \le \gamma R_0^{-(n(m+\lambda)+2)(1+\frac{n}{2})} \iint\limits_{D(R_0)} |u|^{m+\lambda}\mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{45}
$$

By our choice of λ, the integral on the right-hand side of (45) is finite. This completes the proof of Lemma 2.

5 Proof of Theorem 1

Let K be a compact subset in Ω and $\eta \in C_0^{\infty}(\Omega_T)$ such that $\eta = 1$ for $(x,t) \in K \times (0,T)$. Testing (11) by $\varphi = \eta\eta_r,\ \psi = \eta_r$, using (4), the Young inequality, the boundedness of u and passing to the limit $r \to 0$ we get

$$
\operatorname*{ess\,sup}_{0<t<T} \int\limits_{K} u^2(x,t)\mathrm{d}x + \sum_{i=1}^{n} \int\limits_{0}^{t} \int\limits_{K} |u|^{m_i-1} \left|\frac{\partial u}{\partial x_i}\right|^2 \mathrm{d}x\mathrm{d}t \le \alpha. \tag{46}
$$

Testing (11) by $\varphi\eta_r$ where φ is an arbitrary function which belongs to $\overset{o}{V}_m(\Omega_T)$, using estimate (46), and the boundedness of solution, and passing to the limit $r \to 0$ we obtain the integral identity (11) with an arbitrary $\varphi \in \overset{o}{V}_m(\Omega_T)$ and $\psi = 1$. Thus Theorem 1 is proved.

6 Appendix

Lemma 8 [19] *Let* $\{Y_j\},\ j = 0, 1 \ldots$ *be a sequence of positive numbers, satisfying the recursive inequalities*

$$Y_{j+1} \le C b^j Y_j^{1+\alpha}, \tag{47}$$

where $C, b > 1$ *and* $\alpha > 0$ *are given numbers. If*

$$Y_0 \le C^{-\frac{1}{\alpha}} b^{-\frac{1}{\alpha^2}}, \tag{48}$$

then Y_j *converges to zero as* $j \to \infty$.

Lemma 9 [20] *Let* $\Omega \subset R_n, n \ge 3$ *be a bounded domain. Let* ν *be an arbitrary function from the Sobolev space* $W_0^{1,1}$ *such that*

$$\int\limits_{\Omega} \int\limits_{E_R} |\nu|^{\alpha_i} \left| \frac{\partial \nu}{\partial x_i} \right|^{p_i} dx < \infty,$$

where $p_i \ge 1, 1 + \frac{\alpha_i}{p_i} > 0, i = 1, \ldots, n, \sum\limits_{i=1}^{n} \frac{1}{p_i} > 1$. *Then*

$$\nu \in L_{q_*}(\Omega), q_* = \frac{np}{n-p} \left(1 + \frac{1}{n} \sum_{k=1}^{n} \frac{\alpha_k}{p_k} \right) \tag{49}$$

where $\frac{1}{p} = \frac{1}{n} \sum\limits_{i=1}^{n} \frac{1}{p_i}$ *and there exists a positive constant* K *depending only on* $n, \alpha_i, p_i, i = 1, \ldots, n$, *such that the following inequality is valid*

$$\| \nu \|_{L_{q_*}(\Omega)} \le K \prod_{i=1}^{n} \left(\int\limits_{\Omega} |\nu|^{\alpha_i} \left| \frac{\partial \nu}{\partial x_i} \right|^{p_i} dx \right)^{\frac{1}{p_i n \left(1 + \frac{1}{n} \sum\limits_{k=1}^{n} \frac{\alpha_k}{p_k}\right)}} \tag{50}$$

Lemma 10 [16] *Let* $\Omega \subset R_n, n \ge 3$ *be a bounded domain,* $\Omega_T = \Omega \times (0, T), T > 0$. *Let* ν *be an arbitrary function from the Sobolev space* $W_0^{1,1}$ *for every fixed* $0 < t < T$, *and satisfying the following condition*

$$\operatorname*{ess\,sup}_{0<t<T} \int\limits_{\Omega} \nu^2(x,t) dx + \sum_{i=1}^{n} \iint\limits_{\Omega_T} |\nu|^{\alpha_i} \left| \frac{\partial \nu}{\partial x_i} \right|^{p_i} dxdt \le \infty$$

where $p_i \ge 1,\ p_i + \alpha_i > 0, i = 1, \ldots, n, \sum\limits_{i=1}^{n} \frac{1}{p_i} > 1$. *Then*

$$\nu \in L_{\tilde{q}}(\Omega), \tilde{q} = p + \frac{p}{n} \left(\varepsilon + \sum_{k=1}^{n} \frac{\alpha_k}{p_k} \right) \tag{51}$$

Moreover, there exists a positive constant $\tilde{K}$ *depending only on* $n, \alpha_i, p_i, i = 1, \ldots, n$, *and* $\theta \in (0, 1)$, *such that*

$$\| \nu \|_{L_{\tilde{q}}(\Omega_T)} \le \tilde{K} \operatorname*{ess\,sup}_{0<t<T} \|\nu\|_{L_\varepsilon(\Omega)}^{\varepsilon \frac{p}{n}} \sum_{i=1}^{n} \left(\iint\limits_{\Omega_T} |\nu|^{\alpha_i} \left| \frac{\partial \nu}{\partial x_i} \right|^{p_i} dxdt \right)^{\frac{1}{\tilde{q}}}, \tag{52}$$

where the constant ε is defined from the following equality

$$\frac{1}{q} = \frac{\theta}{q_*} + \frac{1-\theta}{\varepsilon}, \;\; \tilde{q} - \varepsilon \frac{p}{n} < q \le q^*. \tag{53}$$

Acknowledgements This work is supported by grant of Ministry of Education and Science of Ukraine (Project Number is 0115 U 000 136). Publication is based on the research provided by the grant support of the State Fund For Fundamental Research (Project Number is 0116 U 007 160).

References

1. Serrin, J.: Isolated singularities of solutions of quasilinear equations. Acta Math. **113**(1), 219–240 (1965)
2. Serrin, J.: Removable singularities of solutions of elliptic equations. Arch. Ration Mech. Anal. **20**(3), 163–169 (1965)
3. Aronson, D.G., Serrin, J.: Local behavior of solutions of quasilinear parabolic equations. Arch. Ration. Mech. Anal. **25**, 81–122 (1967)
4. Kovalevsky, A.A., Skrypnik, I.I., Shishkov, A.E.: Singular solutions of nonlinear elliptic and parabolic equations. In: De Gruyter. Series in Nonlinear Analysis and Applications, Naukova Dumka, Kyiv (2016)
5. Veron, L.: Singularities of solution of second order quasilinear equations. Pitman Research Notes in Mathematics Series, Longman, Harlow (1996)
6. Boccardo, L., Gallouet, T., Marcellini, P.: Anisotropic equations in L^1. Differ. Integral Equ. **9**(1), 209–212 (1996)
7. Cirstea, F.C., Vetois, J.: Fundamental solutions for anisotropic elliptic equations: existence and priori estimates. Commun. Part. Differ. Equ. **40**(4), 727–767 (2015)
8. Liberman, G.: Gradient estimates for anisotropic elliptic equations. Adv. Differ. Equ. **10**(7), 767–812 (2005)
9. Liskevich, V., Skrypnik, I.I.: Hölder countinuity of solutions to an anisotropic elliptic equation. Nonl. Anal. **71**(5–6), 1699–1708 (2009)
10. Marcellini, P.: Regularity and existence of solutions of elliptic equations with $(p, q)-$ growth conditions. J. Differ. Equ. **90**(1), 1–30 (1991)
11. Duzgun, F.G., Marcellini, P., Vespri, V.: Space expansions foe a solution of an anisotropic p-Lapalcian equation by using a parabolic approach. Rivista di Mat.della Univ.di Parma. **5**(1), 93–111 (2014)
12. Giaquinta, M.: Growth conditions and regularity, a counterexample. Manuscr. Math. **59**(2), 245–248 (1987)
13. Marcellini, P.: Un exemple de solution discontinue d'un probleme variationnel dans le cas scalaire, Preprint, Istituto Matematico U. Dini 11, (1987–1988)
14. Fusco, N., Sbordone, C.: Some remarks on the regularity of minima of anisotropic integrals. Commun. PDE **18**(1–2), 153–167 (1993)
15. Kolodij, I.M.: On boundedness of generalized solutions of elliptic differential equations. Vestnik Moskov. Gos. Univ. **5**, 44–52 (1970)
16. Kolodij, I.M.: On boundedness of generalized solutions of parabolic differential equations. Vestnik Moskov. Gos. Univ. **5**, 25–31 (1971)
17. Namlyeyeva, YuV, Shishkov, A.E., Skrypnik, I.I.: Removable isolated singularities for solutions of doubly nonlinear anisotropic parabolic equations. Appl. Anal. **89**(10), 1559–1574 (2010)
18. Namlyeyeva, YuV, Shishkov, A.E., Skrypnik, I.I.: Isolated singularities of solutions of quasilinear anisotripnic elliptic equations. Adv. Nonlin. Stud. **6**(4), 617–641 (2006)
19. Di Benedetto, E.: Degenerate Parabolic Equations. Springer, New York (1993)
20. Zhikov, V.V.: On Lavrentiev's phenomenon. Russ. J. Math. Phys. **3**, 249–269 (1995)